# The ATHENA Class of Risk-Limiting Ballot Polling Audits\*

Filip Zagórski<sup>†1</sup>, Grant McClearn<sup>2</sup>, Sarah Morin<sup>2</sup>, Neal McBurnett, and Poorvi L. Vora<sup>‡2</sup>

<sup>1</sup>Wroclaw University of Science and Technology <sup>2</sup>Department of Computer Science, The George Washington University

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#### **Abstract**

The main risk-limiting ballot polling audit in use today, BRAVO, is designed for use when single ballots are drawn at random and a decision regarding whether to stop the audit or draw another ballot is taken after each ballot draw (ballot-by-ballot (B2) audits). On the other hand, real ballot polling audits draw many ballots in a single round before determining whether to stop (round-by-round (R2) audits). We show that BRAVO results in significant inefficiency when directly applied to real R2 audits. We present the ATHENA class of R2 stopping rules, which we show are risk-limiting if the round schedule is pre-determined (before the audit begins). We prove that each rule is at least as efficient as the corresponding BRAVO stopping rule applied at the end of the round. We have open-source software libraries implementing most of our results.

We show that ATHENA halves the number of ballots required, for all state margins in the 2016 US Presidential election and a first round with 90% stopping probability, when compared to BRAVO (stopping rule applied at the end of the round). We present simulation results supporting the 90% stopping probability claims and our claims for the risk accrued in the first round. Further, ATHENA reduces the number of ballots by more than a quarter for low margins, when compared to the BRAVO stopping rule applied on ballots in selection order. This implies that keeping track of the order when drawing ballots R2 is not beneficial, because ATHENA is more efficient even without information on selection order. These results are significant because current approaches to real ballot polling election audits use the B2 BRAVO rules, requiring about twice as much work on the part of election officials. Applying the rules in selection order requires fewer ballots, but keeping track of the order, and entering it into audit software, adds to the effort.

All our contributions are for audits with zero error of the second kind. Our approach relies on analytical expressions we derive for stopping probabilities. The results of these analytical expressions are verified by comparison with the percentiles Lindeman *et al.* previously obtained using B2 BRAVO simulations [5].

We believe our results may be applied in a straightforward fashion to other SPRTs with  $\beta=0$  if the stopping condition is monotonic increasing with the number of winner ballots (Bayesian audits are an example) but proofs in this paper apply only to BRAVO.

# 1 Introduction

The most popular examples of election tabulation ballot polling audits include BRAVO [5] and Bayesian audits [10]; these audits may be viewed as special cases/extensions of the *sequential probability ratio test (SPRT)*, see [14, 7]. When the decisions of whether to stop the audit or draw more ballots are taken after each ballot draw, and the stopping condition is satisfied exactly when the audit is stopped, these audits—as *SPRTs*—are *most efficient* audits. The term *most efficient* refers here, as elsewhere, to an audit requiring the smallest expected number of ballots given either hypothesis: a correct election outcome or an incorrect one, if the election is drawn from the assumed prior. The

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<sup>†</sup>filip.zagorski@gmail.com, Author was partially supported by Polish National Science Centre contract number DEC-2013/09/D/ST6/03927

<sup>‡</sup>poorvi@gwu.edu

expectation is taken over the randomness of the ballot draws, and, in the case of Bayesian audits, also the randomness of the true tally (Bayesian audits treat the true tally as an unknown random variable).

In real election audits, multiple ballots are drawn in a round before a decision is taken. This paper shows that BRAVO is not a most efficient test in this case, and proposes the ATHENA class of more efficient tests, demonstrating significant decreases in first-round sizes, and proving that the tests are risk-limited if the round schedule is pre-determined (before the audit begins). This could be of consequence for election audits of the 2020 US Presidential election.

#### 1.1 Problem

We refer to audits where decisions are taken after each ballot draw as *ballot-by-ballot* or B2 audits. The general audit, however, is a *round-by-round* or R2 audit where, in the  $j^{th}$  round, some ballots are drawn, after which a decision is taken regarding whether to (a) stop the audit and declare the election outcome correct, (b) stop the audit and go to a manual recount, or (c) draw the  $(j+1)^{th}$  round. A B2 audit is a special case of the R2 audit, when a single ballot is drawn in each round.

There are two ways to apply B2 audit rules to an R2 audit. Consider a total of  $n_j$  ballots drawn after the  $j^{th}$  round, of which  $k_j$  are for the reported winner.

- End-of-round: In this application, the B2 stopping rule for  $k_j$  winner ballots in a sample of  $n_j$  ballots determines whether the audit will stop.
- Selection-ordered-ballots: In this application, ballot order is recorded and the B2 stopping condition is tested  $\forall n \leq n_i$ . The audit stops if the B2 condition is satisfied for any value of  $n \leq n_i$ .

Selection-ordered-ballots is generally more efficient than end-of-round as a means of applying B2 rules to R2 audits, but requires the significant additional effort of preserving enough information to be able to recreate the subtotals of winner ballots in selection order. End-of-round relies only on the tallies and does not require selection order. As our paper shows, neither is a most efficient R2 stopping rule.

One may view the problem we address as lying somewhere between (a) the problem solved by Neyman-Pearson [12]: derive a single-use binary hypothesis testing rule satisfying certain error criteria, and (b) the problem solved by Wald [15]: derive a stopping condition for sequential sampling, satisfying certain error criteria, where the condition is tested draw-by-draw. We address the problem of sequential sampling in rounds, where the condition is tested after multiple draws.

#### 1.2 Our Contributions

Our contributions are as follows:

1. We derive analytical expressions for the risk and probability of stopping, given the history of rounds and the margin for the BRAVO audit. Treating the B2 BRAVO audit as an R2 audit with  $n_j=j$ , we verify that the expressions we derive predict the stopping percentiles originally obtained by Lindeman *et al.* using BRAVO simulations [5, Table 1]. The average of the absolute value of the fractional difference between our results and those of [5] is 0.13%. The largest difference has value 190 ballots, corresponding to a fractional difference of 0.41 %, in the estimate of the expected number of ballots drawn for a margin of 1%. This difference could

be due to small inaccuracies in our computational approach (such as rounding off errors or the finiteness of summations involved in the computations) or the finiteness of the number of simulations used to generate the results of [5]. Our approach is easily extended to audits with stopping conditions that are monotone increasing in the number of ballots for the announced winner, such as Bayesian audits. The code for computing these expressions is available as a MATLAB library, released as open-source under the MIT License [13].

- 2. We present the ATHENA class of R2 stopping rules for audits MINERVA and ATHENA and prove that, if the round schedule is pre-determined (before the audit begins), MINERVA and ATHENA are both risk-limiting and at least as efficient as the corresponding *end-of-round* BRAVO stopping rule. Another audit from the ATHENA class, METIS, is out of scope for this draft.
- 3. We provide experimental results and software to support the use of the proposed audits:
  - To illustrate the efficiency improvements, we compute (without simulations, using the derived analytical expressions), for each state in the 2016 US Presidential election, risk limit  $\alpha=0.1$  and a stopping probability of 0.9, first round sizes for *end-of-round* BRAVO and ATHENA. We find that ATHENA requires about half the number of ballots, across all margins.
  - We compute first round sizes for selection-ordered-ballots BRAVO and find that ATHENA requires about 15-29% fewer ballots for the data of the 2016 US Presidential election, with the improvement being better for smaller margins. Thus ATHENA is more efficient than selection-ordered-ballots BRAVO and does not require the additional book-keeping of recording selection ballot order.
  - We present the results of simulations supporting our predictions of first round stopping probabilities and the risk-limiting properties of ATHENA.
  - Our code for the audits is available as MATLAB and Python libraries [17, 13, 6]. All code is released as open source under the MIT license.

This contribution is important because a number of states have undertaken ballot polling pilots in the last year and plan to use ballot polling audits in November 2020. We hope that our results and code can help developers of auditing software. We also note that, in many scenarios, ballot comparison or batch comparison audits could be more feasible. One may also consider combinations of ballot comparison and ballot polling audits, such as described in [8].

4. The ATHENA class of R2 stopping rules is a class of B2 rules when round size is one. Of theoretical interest, we prove that B2 MINERVA (round size one) has the same stopping rule as B2 BRAVO, as does B2 ATHENA for some values of its parameters.

We do not claim that the audits of the ATHENA class are the most efficient R2 audits with zero error of the second kind ( $\beta = 0$ ). The problem of finding the most efficient R2 audits is open.

Unlike the *SPRT* and other Martingale-based approaches, the stopping rules for audits of the ATHENA class use information about the history of round sizes. For this reason the stopping condition for rounds other than the first one does not depend only on the cumulative sample size and number of winner ballots drawn, but also on the history of individual round sizes.

We do not address some simple extensions of our work in this paper. For example, we do not consider audits with a limit on the total number of ballots drawn in the polling audit (if the audit fails to stop, a full sequential hand count would follow). Were we to do so, we could provide audits with larger stopping probabilities given the same risk limit.

This paper focuses exclusively on BRAVO. However, we expect that our results extend to other risk-limiting SPRTs with  $\beta=0$  and a stopping condition that is monotonic in the number of winner ballots drawn; examples include Bayesian audits.

#### 1.3 Organization

Section 2 presents the model and related work. Section 3 motivates the problem with an example demonstrating that the application of B2 rules to an R2 audit results in inefficiencies. Section 4 introduces the ATHENA class of audits with examples and provides insight into why the audits are risk-limiting and more efficient than either R2 application of B2 BRAVO. Section 5 describes the analytical approaches for computing probabilities for multiple-round audits. Section 6 presents the MINERVA and ATHENA audits, and Section 7 presents rigorous claims of their risk-limiting and efficiency properties in the form of Theorems and Lemmas. Section 8 presents experimental results. Section 9 concludes. Proofs are in the Appendix.

#### 1.4 Acknowledgements

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This version is updated from the previous one to reflect only a couple, and not all, of their valuable suggestions. Importantly, this draft notes that the risk-limiting property is proven only when round sizes are pre-determined. We have also made some of the changes, including some improvements to our notation, and the inclusion of the number of distinct ballots in tables listing the number of ballots required for a first round with 90% stopping probability for 2016 tallies. We plan to soon further update the manuscript to include estimated first round sizes for the 2021 tallies and address all other suggestions made by the reviewers.

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## 2 The Model

We consider a plurality contest and assume ballots are drawn with replacement. We assume all ballots have a vote for either the winner or the loser; because ballots are sampled with replacement, our argument is easily extended to contests with multiple candidates and invalid ballots (as for BRAVO, for example, see [4]). We denote by w the true winner,  $w_a$  the announced winner,  $\ell_a$  the announced loser and p the announced fractional tally for  $w_a$  (typically based on preliminary, uncertified results).

A polling audit will estimate whether  $w_a$  is the true winner. We denote by  $n_j$  the total number of ballots drawn at the end of the  $j^{th}$  round, and by  $k_j$  the corresponding total number of ballots for the winner. Hence the number of new ballots drawn in round j is  $n_j - n_{j-1}$ , and the number of new votes for the winner drawn in round j is  $k_j - k_{j-1}$ . If necessary, one may assume that  $n_0, k_0 = 0$ . We often refer to  $[n_1, n_2, \ldots, n_j, \ldots]$  as the *round schedule*. A B2 audit is an R2 audit with round size  $n_j = j$ . That is, the round schedule of a B2 audit is  $[1, 2, \ldots, j, \ldots]$ .

The total number of ballots drawn at any time during the audit is denoted n (if the number of rounds drawn so far is j,  $n = n_j$ ). The random variable representing the number of ballots drawn so far for the winner is represented by K

(and  $K_j$  to represent the number of ballots drawn for the winner up to the  $j^{th}$  round). We use  $k^*$ ,  $k_*$  and  $\tilde{k}$  to represent specific numbers of winner ballots as well.

The entire sample drawn up to the  $j^{th}$  round, in sequence, forms the *signal* or the *observation*; the corresponding random variable is denoted  $X_j$ , the specific value  $x_j$ . The entire sample drawn so far is denoted X, its specific value x. We do not *a priori* assume a last round for the audit. The audit stops when it satisfies the stopping condition.

#### 2.1 The Model

We model the audit as a binary hypothesis test:

Null hypothesis  $H_0$ : The election outcome is the closest possible incorrect outcome:  $w \neq w_a$  and the fractional vote count for  $w_a$  is  $\frac{1}{2}$ . In particular, if the total number of valid votes is even, the election is a tie. If the total number of valid votes is odd, the margin is one in favor of  $\ell_a$ . In this case, we assume that the number of valid votes is large enough that the fractional vote count is sufficiently close to  $\frac{1}{2}$ . Henceforth, we will refer to both cases as being represented by a fractional vote count of  $\frac{1}{2}$ .

Alternate hypothesis  $H_a$ : The election outcome is correct:  $w = w_a$  and the fractional vote count is as announced.

After each round the test A takes X as input and outputs one of the following:

- Correct: The test estimates that  $w = w_a$  and the audit should stop.
- Incorrect: The test estimates that  $w \neq w_a$ . We stop drawing votes and proceed to perform a complete hand count to determine w.
- *Undetermined (draw more samples):* We need to draw more ballots to improve the estimate.

When the audit stops, it can make one of two kinds of errors:

1. Miss: A miss occurs when  $w \neq w_a$  but the audit misses this, and outputs Correct. We denote by  $P_M$  the probability of a miss:

$$P_M = Pr[\mathcal{A}(X) = Correct \mid H_0]$$

 $P_M$  is the *risk* in risk limiting audits and the Type I error of the test.

2. Unnecessary Hand Count: Similarly, if  $w = w_a$ , but the audit estimates that a hand count must follow, the hand count is unnecessary. We denote the probability of an unnecessary hand count by  $P_U$ :

$$P_U = Pr[\mathcal{A}(X) = Incorrect \mid H_a]$$

 $P_U$  is the Type II error.

Like the BRAVO audit, this paper focuses on tests with  $P_U = 0$ . The risk, on the other hand, is an important (generally) non-zero value characterizing the quality of the audit.

#### 2.2 Related Work

A risk-limiting audit (RLA) with risk limit  $\alpha$ —as described by, for example, Lindeman and Stark [4]—is one for which the risk is smaller than  $\alpha$  for all possible (unknown) true tallies in the election. For convenience when we compare audits, we refer to this audit as an  $\alpha$ -RLA.

**Definition 1** (Risk Limiting Audit ( $\alpha$ -RLA)). *An audit A is a Risk Limiting Audit with risk limit*  $\alpha$  *iff for sample X* 

$$P[\mathcal{A}(X) = Correct \mid H_0] \le \alpha$$

There are many audits that would satisfy the  $\alpha$ -RLA criterion, and not all would be desirable. For example, the constant audit which always outputs *Incorrect* always requires a hand count and is risk-limiting with  $P_M = 0 < \alpha$ ,  $\forall \alpha$ ,  $\forall p$ . However,  $P_U = 1$ , and the audit examines all votes each time; this is undesirable.

An example of an  $\alpha$ -RLA with  $P_U = 0$  and drawing fewer ballots is the B2 BRAVO audit [5] which specifies round size increments of one.

**Definition 2** (BRAVO). An audit A is the B2  $(\alpha, p)$ -BRAVO audit iff the following stopping condition is tested at each ballot draw. If the sample X is of size n and has k ballots for the winner,

$$\mathcal{A}(S) = \begin{cases} Correct & \sigma(k, p, n) \triangleq \frac{p^k (1 - p)^{n - k}}{(\frac{1}{2})^n} \ge \frac{1}{\alpha} \\ Undetermined & else \end{cases}$$
 (1)

Its p-value is  $\sigma(k, p, n)^{-1}$ .

 $\sigma(k, p, n)$  is the *likelihood ratio* of the drawn sequence X. The B2  $(\alpha, p)$ -BRAVO audit is an SPRT [15] with:

 $H_0$ , the null hypothesis: the election is a tie

 $H_a$ , the alternate hypothesis: the fractional tally for the winner is p.

Implicit in Definition 2 is the point that a sequence X is tested only if it has not previously satisfied the test. If  $\mathcal{A}(X_*) = Correct$  for some sequence  $X_*$ , all extensions  $X_*^+$  of  $X_*$  are defined as having passed the test. Determining the stopping condition by evaluating  $\mathcal{A}(X_*^+)$  does not satisfy the assumptions of the test, and the properties of the test do not necessarily apply. As we shall see in Section 3, this is relevant to *end-of-round* BRAVO. In fact, it is relevant to *end-of-round* applications of any B2 audit that is an *SPRT*.

B2 BRAVO is a most efficient test given the hypotheses (if the stopping condition is satisfied exactly every time). Vora shows [14] that B2  $(\alpha, p)$ -BRAVO is an  $\alpha$ -RLA because it assumes a tie for  $H_0$ , which is the wrong election outcome that is hardest to distinguish from the announced one, and hence defines the worst-case risk [14].

Other approaches, such as Rivest's CLIP Audit [9], improve on B2 BRAVO's efficiency subject to certain constraints (namely, of  $\beta$  as defined in [9]).

A prototype of ATHENA mirrored the explicit risk allocation found in Stark's Conservative Statistical Post-Election Audits [11] before ballots are examined for the audit, a list of increasing rounds  $(n_1, n_2, ..., n_j)$ , and a list of corresponding risks  $(\alpha_1, \alpha_2, ..., \alpha_j)$  are generated. Dispensing with auditor flexibility in favor of a predetermined list of rounds and corresponding risks facilitated the investigation of the convolution procedure that underlies a fundamental improvement of ATHENA over BRAVO.

There is a line of work on *group sequential testing* [16, 3, 1, 2] but all results that we were able to find begin with the assumption of a normal distribution and cannot be directly applied to the considered scenario of auditing elections.

#### 3 The Problem

In this section we use an example to illustrate the problems of using B2 rules for an R2 audit.

The B2  $(\alpha, p)$ -Bravo audit, Definition 2, is the following ratio test (inequality (1)) performed after each draw:

$$\sigma(k, p, n) = \frac{p^k (1 - p)^{n - k}}{\left(\frac{1}{2}\right)^n} \ge \frac{1}{\alpha}$$

Because p>1-p and the denominator above does not depend on k,  $\sigma(k,p,n)$  is monotone increasing with k. There is hence a minimum value of k for which the B2  $(\alpha,p)$ -BRAVO stopping condition is satisfied. That is,  $\exists \ k_{min}(\text{BRAVO},n,p,\alpha)$  such that the stopping condition of Definition 2, inequality (1), is:

$$\mathcal{A}(S) = Correct \Leftrightarrow k \geq k_{min}(BRAVO, n, p, \alpha)$$

In fact it is easy to see that  $k_{min}(BRAVO, n, p, \alpha)$  is a discretized straight line as a function of n, with slope and intercept determined by p and  $\alpha$  (see, for example, [15]).

$$k_{min}(\text{BRAVO}, n, p, \alpha) = \lceil m(\text{BRAVO}, p, \alpha) \cdot n + c(\text{BRAVO}, p, \alpha) \rceil$$
 (2)

where

$$\begin{split} m(\text{Bravo}, p, \alpha) &= \frac{\log \frac{\frac{1}{2}}{1-p}}{\log \frac{p}{1-p}} \\ c(\text{Bravo}, p, \alpha) &= -\frac{\log \alpha}{\log \frac{p}{1-p}} \end{split}$$

We drop one or more arguments of  $k_{min}$ , c or m when they are obvious.

**Example 1** (B2 Bravo vs R2 Bravo). Let  $\alpha = 0.1$  and p = 0.75, we get, from equation (2):

$$k_{min}(BRAVO, n, 0.75, 0.1) \approx [0.6309n + 2.0959]$$

Consider ballots drawn in rounds of size 20, 40, 60, ... and the BRAVO condition being tested:

- End-of-Round, which requires a record simply of the tally of the sample polled.
- Selection-ordered-ballots, requires a record of the vote on each ballot polled, in selection order.

Note that the stopping condition is always the BRAVO stopping condition; the variation is in when it is checked.

Figure 1 is a plot of  $k_{min}(BRAVO, n, 0.75, 0.1)$  as a function of round size. It also shows the results of the tests above, performed on an example sequence.

• For a hypothetical sequence, selection-ordered-ballots BRAVO checks the stopping condition at the blue squares till the stopping condition is satisfied, and the audit stops. It has information about the number of ballots for the winner and the total number of ballots drawn at each ballot draw.

• If the same sequence were to go through an end-of-round BRAVO audit, the stopping condition would be checked only at the end of the round, denoted in the figure by black crosses. The audit only has information on vote tallies at the end of the round.

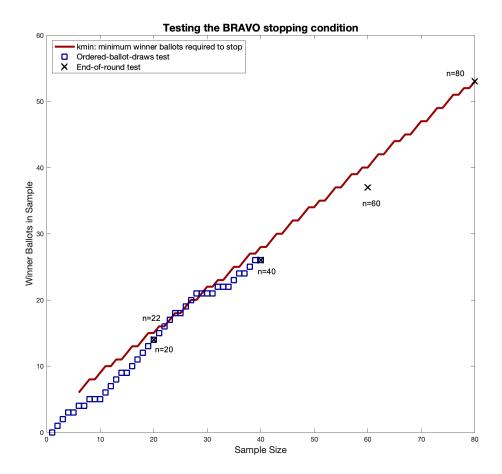


Figure 1: Using BRAVO for a round-by-round audit with p = 0.75,  $\alpha = 0.1$  and round size = 20.

We see that the stopping condition is satisfied during the second round, at n=22, but that it is no longer satisfied when it is tested at the end of that round, at n=40, or the following round, n=60. It is satisfied at the end of the fourth round, n=80, which is the number of ballots drawn in an end-of-round BRAVO audit.

Thus:

- B2 Bravo ends at n = 22, and 22 ballots are drawn.
- End-of-round Bravo ends at n = 80 and 80 ballots are drawn.
- Selection-ordered-ballots Bravo ends at n = 22, and 40 ballots are drawn.

The instance of selection-ordered-ballots BRAVO in our example would stop at the end of the second round after 40 ballots are drawn. Such an audit is risk-limiting even though the condition is not satisfied at the  $40^{th}$  draw. This is

because every time a sequence X satisfies the stopping condition, all extensions of it are defined as having passed the audit as well. In the event that the election outcome is incorrect, any sequence that passes the audit contributes to the risk. A risk-limiting audit ensures that the total risk contribution of all sequences that satisfy the audit is bounded above by the risk limit, whatever the underlying election. This accounting naturally includes risk contributions of all extensions of sequences that pass the audit as well.

Note, however, that *selection-ordered-ballots* discards the extra information contained in the 18 ballots drawn following the 22-ballot draw. It ought to be possible to include this information, obtained at some cost, to better estimate the correctness of the election outcome. (Imagine telling election officials and the public that the p-value of the draw was small enough earlier, that it is not any more, and the math allows us to use the earlier value because if the election outcome is incorrect, it is accounted for in the risk limit). We need not be limited by the B2 BRAVO rules which begin with a large disadvantage when used for R2 audits, as they do not take into account that the ballots are drawn in rounds.

#### 4 An Introduction to the ATHENA Class of Audits

In this section, we use an example to illustrate the workings of a proposed new R2 audit MINERVA. In later sections, we provide more rigorous descriptions of the ATHENA class of R2 audits which we prove are risk-limiting and at least as efficient as *end-of-round* BRAVO. As we mentioned in section 1, our proof requires that the the round schedule be pre-determined (before the audit begins). For example, one may choose a factor a such that  $n_{j+1} - n_j = a(n_j - n_{j-1})$ .

**Example 2** (End-of-Round (0.1, 0.75)-BRAVO). We consider the end-of-round (0.1, 0.75)-BRAVO audit as in the previous section. Denote by  $n_1$  the number of ballots drawn in the first round, and by  $k_1$  those for the winner.

Suppose  $n_1 = 50$ . Figure 2 shows the probability distributions of  $k_1$  for the two hypotheses:

 $H_a$ : the election is as announced, with p = 0.75 (blue solid curve), and

 $H_0$ : the election is a tie (red dashed curve).

We will continue to refer to Figure 2 in the following sections, when we will address the shaded areas.

Suppose  $k_1 = 32$ . The B2 (0.1, 0.75)-BRAVO stopping condition (see inequality(1)) tested end-of-round is:

$$\sigma(k_1, p, n_1) = \frac{p^{k_1} (1 - p)^{n_1 - k_1}}{(\frac{1}{2})^{n_1}} = \frac{\binom{n_1}{k_1} p^{k_1} (1 - p)^{n_1 - k_1}}{\binom{n_1}{k_1} (\frac{1}{2})^{n_1}} = \frac{Pr[k_1 = 32 \mid H_a]}{Pr[k_1 = 32 \mid H_0]} \ge \frac{1}{\alpha}$$
(3)

For our particular example, Figure 2, the likelihood ratio above is:

$$\sigma(32, 0.75, 50) = \frac{Pr[K_1 = 32 \mid H_a]}{Pr[K_1 = 32 \mid H_0]} \approx \frac{0.0264}{0.0160} \approx 1.65 \not\geq \frac{1}{\alpha} = 10$$

And the sample does not pass the end-of-round Bravo audit. Recall that the B2 Bravo p-value is the reciprocal of the above probability ratio. In this example, it is  $\approx 0.6061 > \alpha = 0.1$ .

*This is consistent with the fact that (see Example 1, Section 3):* 

$$32 < k_{min}(BRAVO, 50, 0.75, 0.1) = [0.6309 \cdot 50 + 2.0959] = 34$$
 (4)

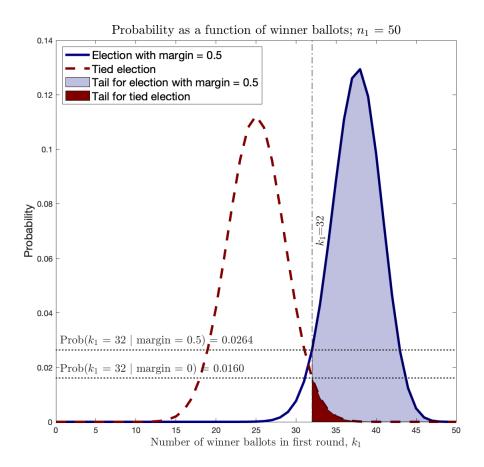


Figure 2: Probability Distribution of Winner Votes for p = 0.75 and  $n_1 = 50$ : First Round.

#### 4.1 The MINERVA Audit

We propose the ATHENA class of audits, which use the tails of the probability distribution functions to define the stopping condition. Here we provide an informal description of the simplest of the ATHENA class, the MINERVA audit.

**Example 3** (The MINERVA Audit). For the parameters of Example 2,  $\alpha = 0.1$ , p = 0.75,  $n_1 = 50$  and  $k_1 = 32$ , we describe the MINERVA stopping condition, a comparison test of the ratio of the tails of the distributions:

$$\tau_1(32, p, n_1) = \frac{Pr[K_1 \ge 32 \mid H_a, n_1]}{Pr[K_1 \ge 32 \mid H_0, n_1]} \ge \frac{1}{\alpha}$$
 (5)

Compare this to the stopping condition for BRAVO, inequality (3).

Note that  $Pr[K_1 \ge 32 \mid H_a]$  is the stopping probability for round 1 (the probability that the audit will stop in round 1 given  $H_a$ ) associated with deciding to stop at  $k_1 = 32$ —and not at smaller values. It is the tail of the solid blue curve, the translucent blue area in Figure 2. Similarly,  $Pr[K_1 = 32 \mid H_0]$  is the associated risk. It is the tail of the red dashed curve denoting the tied election, and shaded red.

For our example, the ratio of the tails of the two curves of Figure 2 is (the values are not denoted in the figure):

$$\tau_1(32, 0.75, 50) = \frac{Pr[K_1 \ge 32 \mid H_a, n_1]}{Pr[K_1 \ge 32 \mid H_0, n_1]} \approx \frac{0.9713}{0.0325} \approx 29.89 > \frac{1}{\alpha} = 10$$

And the sample passes the MINERVA audit.

#### 4.2 The MINERVA audit is risk-limiting

We will prove in Section 7 that the MINERVA stopping condition is monotonic increasing as a function of  $k_1$ ; one may understand this informally as follows. As explained in Section 3,  $\sigma(k_1, p, n_1)$  is monotone increasing with  $k_1$ . The MINERVA ratio  $\tau_1(k_1, p, n_1)$  is a weighted average of the values of  $\sigma(k, p, n_1)$  for  $k \ge k_1$  and is also, hence, monotone increasing with  $k_1$ . If a sample with 32 winner ballots of a total of 50 ballots were to satisfy the stopping condition, so would all samples with  $k_1 \ge 32$ .

Smaller values of  $k_1$  are associated with larger tails in both curves of Figure 2; and the tails denote the stopping probability (given  $H_a$ , the translucent blue tail of the solid blue curve) and the risk (given  $H_0$ , the solid red tail of the dashed red curve). The smaller the value of  $k_1$ , hence, the larger the associated stopping probability and risk. We could simply choose the smallest acceptable  $k_1$  (denoted as  $k_{min,1}$ ) so that the associated risk is  $\alpha$ , but then we could not plan to ever go to another round because we would have exhausted the risk budget in the first round. For the Minerva audit, we choose  $k_1$  so that the risk is no larger than  $\alpha$  times the stopping probability. This allows us to go on to an indefinite number of rounds.

Let  $R_i$  and  $S_i$  be informally defined as follows (more formal definitions follow in Sections 6 and 5):

$$R_j = Pr[MINERVA \text{ audit stops in round } j \text{ and no earlier } | H_0]$$

and

$$S_j = Pr[{\sf MINERVA} \ {\sf audit} \ {\sf stops} \ {\sf in} \ {\sf round} \ j \ {\sf and} \ {\sf no} \ {\sf earlier} \ | \ H_a]$$

We define  $R_j$  and  $S_j$  more carefully in section 7 and describe how to compute these values in section 5. Loosely speaking, they denote the risk associated with the  $j^{th}$  round  $(R_j)$  and the stopping probability of the  $j^{th}$  round  $(S_j)$  respectively.

The MINERVA stopping condition is:

$$\frac{R_j}{S_i} \le \alpha \Rightarrow R_j \le \alpha \cdot S_j$$

If R is the risk of the audit and S its stopping probability,

$$R = \sum_{j} R_{j} \le \alpha \sum_{j} S_{j} \le \alpha \cdot S \le \alpha$$

because S, the stopping probability of the audit, is no larger than 1.

In other words, the total risk of the audit is the sum of the risks of each individual round. The stopping condition ensures that each of these risks is no larger than  $\alpha$  times the corresponding stopping probability. Adding all the risks gives us the total risk, which is no larger than  $\alpha$  times the total stopping probability. Because the total stopping probability cannot be larger than one, the total risk cannot be larger than  $\alpha$ , and MINERVA is risk-limiting.

# 4.3 MINERVA is at least as efficient as end-of-round BRAVO

In this section, we further examine the audit of our previous examples to understand the behavior of the ratios of Bravo and Minerva,  $\sigma(k_1, p, n_1)$  and  $\tau_1(k_1, p, n_1)$  respectively.

**Example 4** (Bravo vs. Minerva Ratios). For the parameters of Examples 2 and 3: p = 0.75,  $\alpha = 0.1$  and  $n_1 = 50$ , Figure 3 presents the likelihood ratio for end-of-round Bravo (green solid line),  $\sigma(k_1, 0.75, 50)$ , and the tail ratio for Minerva (orange dashed line),  $\tau_1(k_1, 0.75, 50)$ , on a log scale. An audit satisfies the stopping condition when its ratio equals or exceeds  $\alpha^{-1} = 10$ .

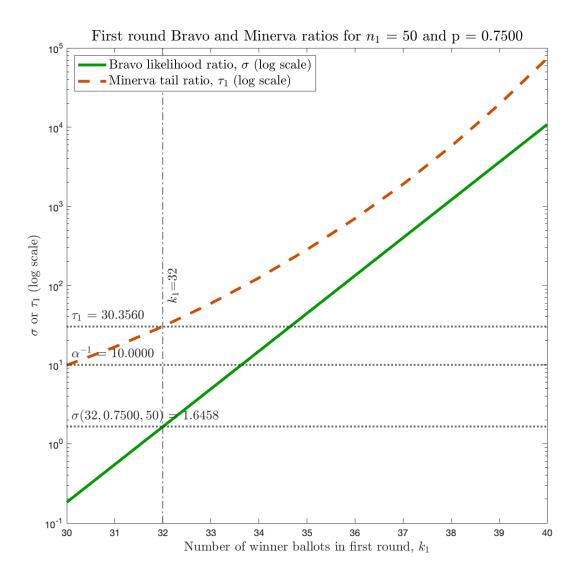


Figure 3: BRAVO and Minerva comparison tests for p = 0.75 and  $n_1 = 50$ : First Round.

We see that

$$\sigma(k_1, 0.75, 50) < \tau_1(k_1, 0.75, 50)$$

This means that any sample satisfying end-of-round BRAVO will also satisfy MINERVA. In fact, it will often be the case that the MINERVA condition will be satisfied and the end-of-round BRAVO one will not.

We have seen earlier (see equation (4), Example 2) that

$$k_{min}(BRAVO, 50, 0.75, 0.1) = 34$$

and end-of-round Bravo will stop for  $k_1 \ge 34$  and no smaller values of  $k_1$ . On the other hand, we see from Figure 3 that Minerva would stop additionally for  $k_1 = 31, 32, 33$ .

The reason for MINERVA stopping at smaller values of  $k_1$  is as follows. Consider  $k_1 = 32$ . While

$$\sigma(32, 0.75, 50) \approx 1.6458 < 10$$

 $\sigma(k_1, 0.75, 50)$  can be much larger for larger values of  $k_1$ . The MINERVA ratio,  $\tau_1$ , at  $k_1=32$  is a weighted average of all the values of  $\sigma(k_1, 0.75, 50)$  for  $k_1 \geq 32$ , allowing the larger values of  $\sigma(k_1, 0.75, 50)$  to "make up" for the smaller ones; in fact,  $\tau_1=30.356$  for  $k_1=32$ .

In other words, because the *end-of-round* BRAVO ratio increases as  $k_1$  increases, the weighted average, the MINERVA ratio, will always be larger than the *end-of-round* BRAVO ratio except if k is the largest possible number of winner votes, in which case the two ratios will be equal. Equivalently, the MINERVA p-value will always be smaller except when k is the largest possible number of winner votes, and the p-values are equal. Thus, MINERVA is at least as efficient.

#### 4.4 The ATHENA audit

In this section we present an example introducing the ATHENA audit.

**Example 5.** We can see from Figure 3 that the MINERVA tail ratio at k = 31 is larger than  $\alpha^{-1}$ . However, the end-of-round BRAVO ratio is smaller than 1:

$$\sigma(31, 0.75, 50) < 1 \Rightarrow Pr[K_1 = 31 \mid H_a] < Pr[K_1 = 31 \mid H_0]$$

which means that the observation  $K_1 = 31$  is more likely given  $H_0$  (the election is a tie) than it is given  $H_a$  (the election is as announced)!

This is technically not an issue; it simply means that MINERVA can stop in such a situation and be risk-limiting. We could also choose to enforce an additional stopping condition of a lower bound on the likelihood ratio of the sample. If the sample x satisfies the MINERVA condition and is of size n with  $k^*$  votes for the winner, the additional stopping condition would be:

$$\frac{Pr[K = k^* \mid H_a]}{Pr[K = k^* \mid H_0]} = \sigma(k^*, p, n) \ge \frac{1}{\delta}$$

for some  $\delta$ . We term this combination of two stopping conditions the ATHENA audit.

A reasonable choice is  $\delta = 1$  (the observation is at least as likely given  $H_a$  as it is given  $H_0$ ). We would, of course, not desire  $\delta < \alpha$ , because we would then be requiring the satisfaction of the BRAVO condition with risk limit  $\delta < \alpha$ .

We can see from Figure 3 that the Athena condition for  $\delta = 1$  is satisfied for  $k^* \geq 32$  and no smaller values of  $k_1$ . Recall that Minerva stops for  $k_1 \geq 31$ . Thus, Minerva would stop for  $k_1 = 31$  and Athena would not.

In our experiments we have observed samples satisfying MINERVA but not ATHENA when the election margin is wide, as in our example. Hence, clearly, ATHENA is not as efficient as MINERVA, because it imposes an additional condition. One may think of MINERVA as determining whether the election outcome is correct, and ATHENA determining, in addition, if the election tally is close enough to the announced tally.

# 5 Computing Risks and Stopping Probabilities for Multiple-Round Audits

In this section we describe how probability distributions may be computed in multiple round audits with monotone stopping conditions; that is, audits where the stopping condition is represented through the use of  $k_{min}$ . We use examples to demonstrate how the probability distributions may be computed for rounds 2 and above.

**Example 6** (Testing the Stopping Condition). Consider an election with p = 0.75 and a risk limit of  $\alpha = 0.1$ . Suppose the first round size is  $n_1 = 50$  and the draw results in  $k_1 = 30$  ballots for the announced winner. Recall that (see equation (4), Example 2)

$$k_{min}(BRAVO, 50, 0.75, 0.1) = 34.$$

and (see Figure 3, Example 4)

$$k_{min}(MINERVA, 50, 0.75, 0.1) = 31$$

Thus the sample passes neither the MINERVA nor the end-of-round BRAVO audit.

Now suppose we draw 50 more ballots to get  $n_2 = 100$  ballots in all, of which  $k_2$  are for the winner. We will need to compute the probability distribution on  $k_2$  to determine the ratio of the tails for the MINERVA stopping condition.

Note that the probability distribution of  $k_2$  is not the binomial distribution for a sample size of 100. In fact, if the audit did not stop in the first round,  $k_1 < 31$  for MINERVA, which means that  $k_2$  can be no larger than 80, even if all 50 ballots in the second round are for the announced winner. Similarly,  $k_2$  for the end-of-round BRAVO audit can be no larger than 83.

If the audit continues, the maximum number of ballots before new ones are drawn is 33 for Bravo and 30 for Minerva. The probability distributions before the new sample is drawn are as shown in Figures 4 and 5, and may be denoted as:

$$Pr[k_1 \wedge (\mathcal{A}_{\mathcal{B}}(X_1) \neq Correct) \mid H_a], Pr[k_1 \wedge (\mathcal{A}_{\mathcal{B}}(X_1) \neq Correct) \mid H_0]$$

and

$$Pr[k_1 \wedge (\mathcal{A}_{\mathcal{M}}(X_1) \neq Correct) \mid H_a], \ Pr[k_1 \wedge (\mathcal{A}_{\mathcal{M}}(X_1) \neq Correct) \mid H_0]$$

where  $A_B$  and  $A_M$  denote the end-of-round Bravo and Minerva audits for the given parameters.

The "discarded" tails, in both cases, represent the probabilities that the audit stops. When this is conditional on  $H_a$ , we refer to it as the stopping probability of the round  $(S_1)$ , large values are good. When it is conditional on  $H_0$ , it is the worst-case risk corresponding to the round  $(R_1)$ , large values are bad. Recall that our stopping condition bounds the worst-case risk for the round to be no larger than a fraction  $\alpha$  of the stopping probability.

Using the above probability distributions, we can now compute the distribution of ballots for the announced winner in the sample of size 100, which we obtain after drawing 50 more ballots.

**Example 7** (Second Round Distribution). Continuing with Example 6, we consider an election with p = 0.75, risk limit  $\alpha = 0.1$  and round sizes  $n_1 = 50$ ,  $n_2 = 100$ . We wish to compute the probability distribution for  $K_2$ , the number

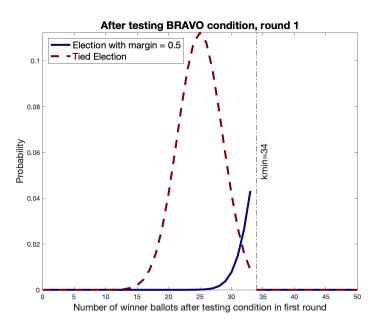


Figure 4: Probability Distribution of Winner Ballots for *end-of-round* BRAVO: p = 0.75,  $n_1 = 50$ : After Testing the Stopping Condition for the First Round.

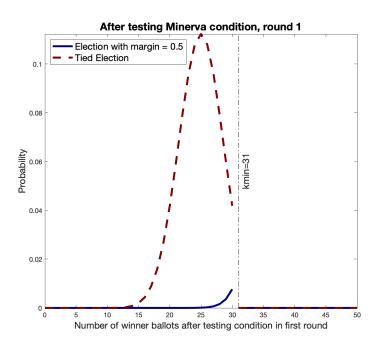


Figure 5: Probability Distribution of Winner Ballots for MINERVA: p = 0.75,  $n_1 = 50$ : After Testing the Stopping Condition for the First Round.

of votes for the announced winner after drawing the second round of ballots. Recall that (see equation (4), Example 2)

$$k_{min}(BRAVO, 50, 0.75, 0.1) = 34.$$

and (see Figure 3, Example 4)

$$k_{min}(Minerva, 50, 0.75, 0.1) = 31$$

First consider the end-of-round BRAVO audit. After the first round stopping condition is tested, and the audit stopped if the condition is satisfied, the number of votes for the winner is at most 33. Let  $K_1$  be the number of votes for the winner. Then  $K_1$  lies between 0 and 33. It is distributed as in Figure 4, and we denote the distribution by  $f(k_1 \mid H_0)$  for the null hypothesis (tied election, represented by the red dashed line) and  $f(k_1 \mid H_a)$  for the alternate hypothesis (election is as announced, represented by the blue solid line).

There would be a total of  $K_2 = k_2$  winner ballots in the sample after the second draw if  $k_2 - k_1$  winner ballots were drawn among the 50 new ballots drawn in round 2.  $k_2 - k_1$  is a random variable, and its distribution is the binomial distribution for the draw of size 50.

If we denote the distribution of  $K_2$  as g, it is:

$$g(k_2 \mid H) = \sum_{k_1 = \max\{0, k_2 - 50\}}^{\min\{k_{\min} - 1, k_2\}} f(k_1 \mid H) \cdot binomial(k_2 - k_1, 50, H)$$

where binomial(j, n, H) is the probability of drawing j votes for the announced winner in a sample of size n, when the fractional vote for the announced winner is  $\frac{1}{2}$  for  $H = H_0$  and p for  $H = H_a$ .

The above expression is known as the convolution of the two functions, and is denoted:

$$g(\cdot \mid H) = f(\cdot \mid H) \circledast binomial(., 50, H)$$

where  $\circledast$  represents the convolution operator and H the hypothesis. The convolution of two functions can be computed efficiently using Fourier Transforms; this result is the convolution theorem.

After drawing the second sample, the probability distributions for BRAVO are as in Figure 6.

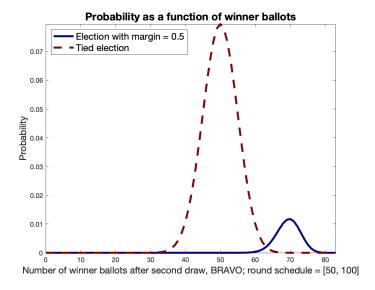


Figure 6: Probability Distribution of Winner Ballots for *end-of-round* BRAVO: p = 0.75,  $n_1 = 50$ ,  $n_2 = 100$ : After Drawing the Second Round.

Similarly, for MINERVA,  $k_1$  lies between 0 and 30.  $k_2$  is similarly the convolution of the function(s) represented in Figure 5 and the binomial distribution corresponding to a draw of 50 ballots for the respective hypotheses. After drawing the second sample, the probability distributions for MINERVA are as in Figure 7.

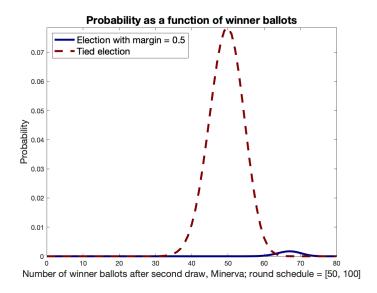


Figure 7: Probability Distribution of Winner Ballots for MINERVA: p = 0.75,  $n_1 = 50$ ,  $n_2 = 100$ : After Drawing the Second Round.

In order to compute probability distributions for the next round, we would first compute the value of  $k_{min,1}$  for this round using the tail ratio, then zero the probability distributions for the value of  $k_{min,1}$  and above, and then perform a convolution with the binomial distribution corresponding to the size of the next draw. And so on.

Probability distributions for B2 audits may be computed similarly, with the round schedule: (1, 2, ..., i, ...). We used this approach to compute percentiles for the BRAVO stopping probabilities; see Section 8 for the results.

#### 6 The ATHENA Class of Audits

In this section we rigorously describe MINERVA and ATHENA, two audits from the new ATHENA class of risk-limiting audits. The stopping condition for BRAVO is a comparison test for the ratio of probabilities of the number of winner ballots. On the other hand, the stopping conditions for the ATHENA class are comparison tests for the ratio of the complementary cumulative distribution functions (cdfs). For the ATHENA class of audits, the stopping condition for a given round does depend on previous round sizes, which are required to compute the complementary cdfs, but not on future round sizes.

#### 6.1 The MINERVA audit

Given the B2  $(\alpha, p)$ -BRAVO test we define the corresponding R2 MINERVA test by its stopping condition, which is a comparison test of the ratio of the complementary cdfs of samples that did not satisfy the stopping condition for any previous round. We expect that similar R2 MINERVA tests can be defined for other *SPRT*s with zero error of the second

kind whose probability ratio is a monotonic increasing function of k, such as Bayesian audits, but do not address these in this paper. Note that the round schedule is predetermined before the audit begins.

**Definition 3**  $((\alpha, p, (n_1, n_2, \dots, n_j, \dots))$ -MINERVA). Given B2  $(\alpha, p)$ -BRAVO and round sizes  $n_1, n_2, \dots, n_j, \dots$ , the corresponding R2 MINERVA stopping rule for the  $(j+1)^{th}$  round is:

$$\mathcal{A}(X_{j+1}) = \begin{cases} Correct & \tau_{j+1}(k_{j+1}, p, (n_1, n_2, \dots, n_j, n_{j+1}), \alpha) \ge \frac{1}{\alpha} \\ Undetermined (draw more samples) & else \end{cases}$$

$$(6)$$

where 
$$\tau_{j+1}$$
 is the complementary cumulative distribution ratio for the  $j+1^{th}$  round: 
$$\tau_{j+1}(k_{j+1},p,(n_1,n_2,\ldots,n_j,n_{j+1}),\alpha) = \frac{Pr[K_{j+1} \geq k_{j+1} \land \forall_{i \leq j} (\mathcal{A}(X_i) \neq Correct) \mid H_a,n_{j+1}]}{Pr[K_{j+1} \geq k_{j+1} \land \forall_{i \leq j} (\mathcal{A}(X_i) \neq Correct) \mid tie,n_{j+1}]} \quad j \geq 1 \quad (7)$$

and, as with B2  $(\alpha, p)$ -BRAVO,  $H_a$ , the alternate hypothesis, is that the fractional tally for the winner is p

Clearly, for j = 0 and the first round,

$$\tau_1(k_1, p, n_1) = \frac{Pr[K_1 \ge k_1 \mid H_a, n_1]}{Pr[K_1 \ge k_1 \mid tie, n_1]}$$

#### The ATHENA audit 6.2

In addition to comparing the ratio of complementary cumulative distribution functions as in MINERVA, the ATHENA audit also enforces a lower bound on the *probability ratio*,  $\sigma$ , of the B2  $(\alpha, p)$ -BRAVO test.

**Definition 4**  $((\alpha, \delta, p, (n_1, n_2, \dots, n_j, \dots))$ -ATHENA). Given B2  $(\alpha, p)$ -BRAVO, round sizes  $n_1, n_2, \dots, n_j, \dots$  and parameter  $\delta$ , the corresponding ATHENA stopping rule for the  $(j+1)^{th}$  round is:

$$\mathcal{A}(X_{j+1}) = \begin{cases} Correct & \omega_{j+1}(k_{j+1}, p, (n_1, n_2, \dots, n_j, n_{j+1})) \ge \frac{1}{\alpha} \\ & \wedge \sigma(k_{j+1}, p, n_{j+1}) \ge \frac{1}{\delta} \end{cases}$$

$$Undetermined (draw more samples) \quad else$$
(8)

where  $\omega_{j+1}$  is the complementary cumulative distribution ratio for the  $(j+1)^{th}$  round:

$$\omega_{j+1}(k_{j+1}, p, (n_1, n_2, \dots, n_j, n_{j+1}), \alpha, \delta) = \frac{Pr[K_{j+1} \ge k_{j+1} \land \forall_{i \le j} (\mathcal{A}(S_i) \ne Correct) \mid H_a, n_{j+1}]}{Pr[K_{j+1} \ge k_{j+1} \land \forall_{i \le j} (\mathcal{A}(X_i) \ne Correct) \mid tie, n_{j+1}]} \quad j \ge 1$$
(9)

$$\sigma(k_{j+1}, p, n_{j+1}) = \frac{p^{k_{j+1}} (1-p)^{n_{j+1} - k_{j+1}}}{\left(\frac{1}{2}\right)^{n_{j+1}}} \quad j \ge 1$$

and, as with  $B2(\alpha, p)$ -BRAVO,  $H_a$ , the alternate hypothesis, is that the fractional tally for the winner is p.

Clearly, for j = 0 and the first round,

$$\omega_1(k_1, p, n_1) = \frac{Pr[K_1 \ge k_1 \mid H_a, n_1]}{Pr[K_1 \ge k_1 \mid tie, n_1]}$$

We further define the risk and stopping probability associated with each round.

**Definition 5**  $(S_j)$ . The probability of stopping in the  $j^{th}$  round for audit A is defined as:

$$S_j = Pr[(\mathcal{A}(X_j) = Correct) \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_a, n_j]$$

**Definition 6**  $(R_i)$ . The risk of the  $j^{th}$  round of audit A is defined as:

$$R_j = Pr[(\mathcal{A}(X_j) = Correct) \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_0, n_j]$$

# 7 Risk-Limiting Properties of the ATHENA Class of Audits

In this section we present the risk-limiting and efficiency properties of MINERVA and ATHENA. We begin with an outline of our approach. Rigorous statements follow and proofs are in the Appendix.

#### 7.1 An outline of the proofs

In this section we provide an outline of the claims and proofs.

Using induction on the number of rounds, we prove a couple of interesting properties, including, at the core, that the likelihood ratio of  $K_j$  (total number of winner ballots) in the  $j^{th}$  round is

$$\frac{Pr[K_j = k \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_a, n_j]}{Pr[K_j = k \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_0, n_j]} = \sigma(k, p, n) \triangleq \frac{p^k (1 - p)^{n - k}}{(\frac{1}{2})^n}$$

That is, for k winner ballots in round j, when sequences are restricted to those that did not satisfy stopping conditions in previous rounds, the likelihood ratio is simply  $\sigma(k, p, n)$ , independent of any additional constraints on sequence order and the past or future round schedule. This property leads to the result that the test in the  $j^{th}$  round is a comparison test for  $k_j$ .

For the base case, it is easily shown that the likelihood ratio in the first round is  $\sigma(k, p, n)$ , as there is no previous round:

$$\frac{Pr[K_1 = k \mid H_a, n]}{Pr[K_1 = k \mid H_0, n]} = \frac{\binom{n}{k} p^k (1 - p)^{n - k}}{\binom{n}{k} (\frac{1}{2})^n} = \sigma(k, p, n)$$

 $\sigma(k,p,n)$  is easily seen to be monotone increasing with k because  $p>\frac{1}{2}$ .

The induction step proceeds as follows. Suppose the likelihood ratio for the number of winner ballots in round j is  $\sigma(k,p,n)$ . The ratios used for the stopping conditions of the  $j^{th}$  round in MINERVA and ATHENA

$$\tau_i(k, p, (n_1, n_2, \dots, n_i), \alpha)$$

and

$$\omega_i(k, p, (n_1, n_2, \dots, n_i), \alpha, \delta)$$

respectively, are weighted sums of  $\sigma(k_j, p, n)$  for  $k_j \ge k$ . Because  $\sigma(k_j, p, n)$  is monotone increasing with  $k_j$ , the respective stopping conditions are monotone increasing with k, and can be expressed as comparison tests for k.

ATHENA has two conditions. The second one is a comparison test for the likelihood ratio, and hence also equivalent to a comparison test for the number of winner ballots. The overall comparison test is the stricter of the two, and is also a comparison test. Thus the stopping condition for round j is a test of the form  $k \ge k_{min,j}$  for a  $k_{min,j}$  that depends

on the audit, its parameters including previous round sizes, the election parameters and the risk limit. As described in section 4, one can use convolution to compute the probability distributions for  $k_{j+1}$ .

To determine the nature of the likelihood ratio for round j+1, we proceed as follows. The likelihood ratio for k in round j is assumed to be  $\sigma(k,p,n)$ . Hence, the distribution of k in round j given  $H_a$  is a multiple of  $p^k(1-p)^{n-k}$ , and that given  $H_0$  is the same multiple of  $(\frac{1}{2})^n$ . The multiplying factor itself is a function of k, current and previous round sizes, p and  $\alpha$ , as it captures the previous comparison tests on the number of winner ballots. On convolution, when one obtains the distributions for round j+1, the multiplying factors change, but the one for the distribution given  $H_a$  is the same as that for the distribution given  $H_0$ . Loosely speaking, the number of ways of obtaining a sequence with k winner ballots in a round of size n, given previous round sizes (and hence previous comparison tests for winner ballots), is independent of the hypothesis. Thus the likelihood ratio of k in round j+1 remains  $\sigma(k,p,n)$ .

It is now easy to show that the audits are risk-limiting. The tail (beginning at  $k_{min,j}$ ) of the probability distribution of  $K_j$  in the  $j^{th}$  round conditional on hypothesis  $H_a$  ( $H_0$ ) is the stopping probability (risk) associated with the  $j^{th}$  round. Both MINERVA and ATHENA audits require that the tail corresponding to  $H_0$  (the risk corresponding to round j) be no more than  $\alpha$  times the tail corresponding to  $H_a$  (the stopping probability), thus ensuring that the sum of all the risk contributions of the rounds is no more than  $\alpha$  times the total stopping probability, and hence no more than  $\alpha$ .

Both MINERVA and ATHENA are more efficient than end-of-round BRAVO. This is because the ratios  $\sigma$ ,  $\tau_j$  and  $\omega_j$  are all compared to the same value,  $\frac{1}{\alpha}$ . For k winner ballots, the ratio for BRAVO is  $\sigma(k,p,n)$ , which is monotone increasing with k. The ratios for ATHENA and MINERVA,  $\omega_j$  and  $\tau_j$  respectively, are weighted sums of  $\sigma(k_j,p,n)$  for  $k_j \geq k$  and hence strictly larger, unless k is the largest value with non-zero probability (it would be value of of  $k_{min}-1$  for the previous round, plus the size of the current draw), when they are equal. Thus, if a sample satisfies the end-of-round BRAVO condition, it also satisfies the conditions on  $\omega_j$  and  $\tau_j$ . The ATHENA audit includes a second test, which is a comparison test for  $\sigma(k,p,n)$ . If  $\delta \geq \alpha$ , this too is satisfied if the end-of-round BRAVO condition is satisfied.

We state the above claims more formally in the rest of this section, and prove them in the Appendix. We also prove that the MINERVA and ATHENA ( $\delta \geq \alpha$ ) stopping conditions, when round increments are specified to be one, are equivalent to the B2 BRAVO condition, though p-values are not the same except for samples where the audits stop.

#### 7.2 Notation

We establish some shorthand notation which will be useful.

For ease of notation, when the audit and its parameters: round schedule  $(n_1, n_2, ...)$ , risk limit  $\alpha$ , fractional vote for the winner  $w_a$  are fixed, we denote:

$$S_{j}(k_{j}) \triangleq Pr[K_{j} \geq k_{j} \land \forall_{i < j} (\mathcal{A}(X_{i}) \neq Correct) \mid H_{a}, n_{1}, \dots, n_{j}]$$
  
$$R_{j}(k_{j}) \triangleq Pr[K_{j} \geq k_{j} \land \forall_{i < j} (\mathcal{A}(X_{i}) \neq Correct) \mid H_{0}, n_{1}, \dots, n_{j}]$$

Thus  $\frac{S_j(k_j)}{R_j(k_j)}$  is the ratio of the complementary cdfs in round j when the number of winner ballots drawn is  $k_j$ , and the sequence did not satisfy the stopping condition in a previous round.

Similarly,

$$s_j(k_j) \triangleq Pr[K_j = k_j \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_a, n_1, \dots, n_j]$$

and

$$r_j(k_j) \triangleq Pr[K_j = k_j \land \forall_{i < j} (\mathcal{A}(X_i) \neq Correct) \mid H_0, n_1, \dots, n_j]$$

and  $\frac{s_j(k_j)}{r_j(k_j)}$  is the likelihood ratio of  $k_j$  winner ballots in round j when the sequence did not satisfy the stopping condition in a previous round.

Note also the following simple observation:

$$S_j(k_j) = \sum_{k=k_j}^{n_j} s_j(k)$$

$$R_j(k_j) = \sum_{k=k_j}^{n_j} r_j(k)$$

$$(10)$$

Recall that, when we do not refer to parameters at all,  $S_j$  corresponds to the stopping probability of the  $j^{th}$  round and is not a function of the sample drawn, but of the audit. Similarly for  $R_j$ , R and S. (See Definitions 5 and 6).

#### 7.3 Properties of the MINERVA and ATHENA complementary cdf ratios

In this section we prove interesting properties of the MINERVA and ATHENA ratios that are necessary to prove that the audits are risk-limiting.

Note that the B2  $(\alpha, p)$ -BRAVO stopping condition is based on  $\sigma(k, p, n)$ :

$$\sigma(k, p, n) = \frac{p^k (1-p)^{n-k}}{(\frac{1}{2})^n}$$

where the history of round size is completely captured in the total number of ballots drawn, n.

We prove that  $\sigma(k, p, n)$  is also the likelihood ratio of winner ballots in all rounds of the MINERVA and ATHENA audits, even though round sizes are not constrained in any way. We additionally prove other interesting properties.

**Theorem 1.** For the  $(\alpha, p, (n_1, n_2, ..., n_j, ...))$ -MINERVA test, if the round schedule is pre-determined (before the audit begins), the following are true for j = 1, 2, 3, ...

I.  $\frac{s_j(k_j)}{r_j(k_j)} = \sigma(k_j, p, n_j)$ 

when  $r_i(k_i)$  and  $s_i(k_i)$  are defined and non-zero.

- 2.  $\tau_j(k_j, p, (n_1, n_2, \dots, n_j), \alpha)$  is monotone increasing as a function of  $k_j$ .
- 3.  $\exists k_{min,j}(MINERVA, (n_1, n_2, \dots, n_j), p, \alpha)$  such that

$$\mathcal{A}(X_j) \ = Correct \Leftrightarrow K_j \geq k_{min,j}(\texttt{Minerva}, (n_1, n_2, \dots, n_j), p, \alpha)$$

Similarly:

**Theorem 2.** For the  $(\alpha, \delta, p, (n_1, n_2, \dots, n_j, \dots))$ -ATHENA test, if the round schedule is pre-determined (before the audit begins), the following are true for  $j = 1, 2, 3, \dots$ :

1. 
$$\frac{s_j(k_j)}{r_j(k_j)}=\sigma(k_j,p,n_j)$$
 when  $r_j(k_j)$  and  $s_j(k_j)$  are defined and non-zero.

- 2.  $\omega_j(k_j, p, (n_1, n_2, \dots, n_j), \alpha, \delta)$  is monotone increasing as a function of  $k_j$ .
- 3.  $\exists k_{min,j}(ATHENA, (n_1, n_2, \dots, n_j), p, \alpha, \delta)$  such that

$$\mathcal{A}(X_i) = Correct \Leftrightarrow K_i \geq k_{min,i}(ATHENA, (n_1, n_2, \dots, n_i), p, \alpha)$$

#### 7.4 MINERVA and ATHENA are risk-limiting

Now we may state the results on the risk limiting properties of the audits.

**Theorem 3.**  $(\alpha, H_a, (n_1, n_2, \dots, n_j, \dots))$ -MINERVA is an  $\alpha$ -RLA if the round schedule is pre-determined (before the audit begins).

Exactly the same approach may be used to prove:

**Theorem 4.**  $(\alpha, \delta, H_a, (n_1, n_2, \dots, n_j, \dots))$ -ATHENA is an  $\alpha$ -RLA if the round schedule is pre-determined (before the audit begins).

#### 7.5 Properties of B2 versions of MINERVA and ATHENA

In this section we study the relationship between B2 BRAVO and MINERVA with each round consisting of a single ballot draw. We make the following observation: samples satisfying the stopping condition of  $(\alpha, p)$ -BRAVO, performed ballot-by-ballot, are exactly those satisfying that of the  $(\alpha, H_a, (1, 2, 3, \ldots, j, \ldots))$ -MINERVA audit, where  $H_a$  is the hypothesis that the winner's fractional tally is p. The p-values of the two audits, however, differ except at their values of  $k_{min}$ .

**Theorem 5.** The B2  $(\alpha, p)$ -Bravo audit stops for a sample of size  $n_j$  with  $k_j$  ballots for the winner, if and only if the  $(\alpha, H_a, (1, 2, 3, \ldots, j, \ldots))$ -MINERVA audit stops.

**Corollary 1.** Given  $\alpha, p, \delta$  such that  $\delta \geq \alpha$ , the B2  $(\alpha, p)$ -Bravo audit stops for a sample of size  $n_j$  with  $k_j$  ballots for the winner, if and only if the  $(\alpha, \delta, H_a, (1, 2, 3, \ldots, j, \ldots))$ -Athena audit stops.

## 7.6 Strong RLAs

We define a new audit property which characterizes the difference between MINERVA and ATHENA.

**Definition 7** (Strong Risk Limiting Audit  $(\alpha, \delta)$ -RLA). *An audit procedure*  $\mathcal{A}$  *is an*  $(\alpha, \delta)$ -Strong Risk Limiting Audit if it is a Risk Limiting Audit with risk level  $\alpha$  *and, if, for every accepted sample, the likelihood-ratio is bounded below* by  $\frac{1}{\delta}$ :

$$\mathcal{A}(X) = correct \Rightarrow \frac{Pr[X \mid H_a]}{Pr[X \mid H_0]} \ge \frac{1}{\delta}$$

It is easy to see that:

**Lemma 1.** B2 BRAVO, end-of-round BRAVO and selection-ordered-ballots BRAVO are  $(\alpha, \alpha)$ -strong RLAs.

#### 7.7 Efficiency

In this section we present an efficiency result for MINERVA and ATHENA (for  $\delta \geq \alpha$ ).

**Theorem 6.** Given sample X of size  $n_i$  with  $k_i$  samples for the winner,

$$A_B(X) = Correct \Rightarrow A_A(X) = Correct$$

where  $A_B$  denotes the  $(\alpha, p)$ -Bravo test and  $A_A$  the  $(\alpha, p, (n_1, n_2, \ldots, n_j, \ldots))$ -Minerva test or the  $(\alpha, \delta, p, (n_1, n_2, \ldots, n_j, \ldots))$ -Athena test for  $\delta \geq \alpha$  if the round schedule is pre-determined (before the audit begins).

From this it follows that MINERVA and ATHENA (for  $\delta \geq \alpha$ ) is each at least as efficient as the corresponding *end-of-round* application of B2 rules. In section 8.2 we demonstrate that ATHENA and MINERVA can be considerably more efficient.

# 8 Experimental Results

In this section we describe our experimental results. We first present our analytical results for percentiles of the BRAVO stopping condition, and compare them with those reported by Lindeman *et al.* [5, Table 1]. We then describe our estimates of first round sizes, comparing ATHENA ( $\delta=1$ ) to both *end-of-round* BRAVO and *selection-ordered-ballots* BRAVO. Finally, we present simulation results.

#### 8.1 B2 BRAVO Percentile Verification

We used the approach described in Section 5 to generate the probability distributions for B2 BRAVO using various election margins to see how our estimates compared to those obtained by Lindeman *et al.* [5, Table 1]. They used 10,000 simulations.

Table 1 presents our values. Values in parentheses are from [5, Table 1], where they differ. Also listed in the table is Average Sample Number (ASN), which is computed using a standard theoretical estimate (and not using our analytical expressions, nor simulations). It provides a baseline to compare with the values for the Expected Ballots column. Some of the difference between our values and those of [5, Table 1] is likely due to rounding off. Further, we notice that both our values and those of [5, Table 1], when they differ from ASN, are lower than ASN. In our case, the difference is likely due to the fact that we compute our probability distributions for only up to 6ASN draws, using a finite summation to estimate the probability distributions, and we model the discrete character of the problem, which is not captured by ASN. The largest difference between our values and those of [5, Table 1] is 190 ballots, corresponding to a fractional difference of 0.41 %, in the estimate of the expected number of ballots drawn for a margin of 1%. Our value is further from ASN. The average of the absolute value of the fractional difference between our results and those of [5] is 0.13%. The differences between our values and those obtained with simulations could be because 10,000 simulations may not be sufficiently accurate at the lower margins, where most of the errors are. It could also be because our finite summation is not sufficient at low margin.

Margin	$25^{th}$	$50^{th}$	75 <sup>th</sup>	90 <sup>th</sup>	$99^{th}$	Expected Ballots	ASN
0.4	12	22	38	60	131	29.47	30.03
						(30)	
0.3	23	38	66	108	236	52.83	53.25
						(53)	
0.2	49	84	149	244	538	118.00	118.88
						(119)	
0.18	77	131	231	381	842	183.60	184.89
					(840)	(184)	
0.1	193	332	587	974	2,155	466.47	469.26
					(2,157)	(469)	
0.08	301	518	916	1,520	3,366	726.95	730.80
						(730)	
0.06	531	914	1,621	2,698	5,976	1,287.60	1,294.62
			(1,619)	(2,700)	(5,980)	(1,294)	
0.04	1,190	2,051	3,637	6,055	13,433	2,887.28	2,901.97
	(1,188)			(6,053)	(13,455)	(2,900)	
0.02	4,727	8,161	14,493	24,155	53,646	11,506.45	11,561.66
	(4,725)	(8,157)	(14,486)	(24,149)	(53,640)	(11,556)	
0.01	18,845	32,566	57,856	96,469	214,385	45,935.85	46,150.44
	(18,839)	(32,547)	(57,838)	(96,411)	(214,491)	(46,126)	

Table 1: Computed Estimates of B2 BRAVO Stopping Probabilities. Values in parentheses are those from [5, Table 1] that differ.

#### 8.2 First-round estimates

In this section we report the results of our estimates for first round sizes for 90% stopping probability for both *end-of-round* BRAVO and ATHENA ( $\delta=1$ ), for the announced statewide results of the 2016 US Presidential election. Our results are presented in Table 2.

We constructed a table of stopping probability as a function of round size for a given margin, where the stopping probability of a round is the tail corresponding to the  $k_{min}$  value for that round size. We observed that the stopping probability is not a monotone increasing function of round size. This is because, if  $k_{min}$  increases with round size (it does not decrease, but it may remain the same), the stopping probability may decrease slightly. For our first-round-size computations reported in section 8, we use the more conservative estimates: given a desired stopping probability  $\rho$ , we chose round sizes such that all larger rounds stopped with probability at least  $\rho$ . For small margins, smaller than 0.05 (except the states of Michigan and New Hampshire, which had the smallest margin), we did not construct the entire table, but began looking for the values by checking if the values of k with the requisite tail size satisfied the stopping condition. Finally, for the states of Michigan and New Hampshire, we approximated round size by estimating the binomial as a gaussian.

We first examined the relationship between end-of-round BRAVO and ATHENA ( $\delta=1$ ) first-round sizes. We estimated the stopping probability and the first-round sizes for end-of-round BRAVO as described above. We use the round size beyond which the stopping probability is at least 90% for both end-of-round BRAVO and ATHENA, thus our round-size estimates are conservative. We scaled these estimates by the ratio of total ballots cast to the number of valid ballots in the contest between the two leading candidates, Trump and Clinton. Better estimates would result from taking into consideration every possible margin for every round size. Our approach, however, is sufficient for the purposes of a rough comparison (we have developed software for the more accurate approach; it is being tested). Of course, some

of these sizes are too large for consideration in a real audit; in particular, the round size for New Hampshire is more than the number of ballots cast in the election.

It is noteworthy that, across all margins, ATHENA first round sizes are about half those of *end-of-round* BRAVO. We note that the number of distinct ballots drawn (thanks to Philip B. Stark for the suggestion) behaves similarly with margin, except for the smallest margins, when the number of ballots drawn is so large that the number of distinct ballots drawn differs from the number of ballot draws.

We also estimate first round sizes for 90% stopping probability for *selection-ordered-ballots* Bravo by treating it as a multiple-round audit. We use the approach described in Section 5, for which our verification results were presented in Section 8.1. Our results are presented in Table 3. We currently omit estimates for states with margins smaller than 0.01. In the other states, we observe that the improvement on using ATHENA ( $\delta=1$ ) is 15%-29%, with greater improvements for smaller margins. Recall that, unlike *selection-ordered-ballots* Bravo, ATHENA does not require that the ballots be noted in selection order; sample tallies are sufficient.

We present the same data (number of ballot draws, not number of distinct ballots) in the form of plots. Figure 8 plots the first round sizes of ATHENA, end-of-round BRAVO and selection-ordered-ballots BRAVO on a log scale as a function of margin. One can observe that the ATHENA round sizes are the smallest, and the end-of-round the largest. Figure 9 plots the ATHENA round size as a fraction of the corresponding end-of-round BRAVO and selection-ordered-ballots BRAVO round sizes. There is a small variation with margin, with the ATHENA round sizes being smaller fractions for smaller margins (that is, the improvement from using ATHENA is larger for smaller margins). Note that a couple of states with the largest margins do not have the largest ratios. This is likely because the round sizes are very small, and hence a difference of a single ballot changes the ratio considerably.

Alabama Alaska Arizona Arkansas California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia Hawaii	0.2875 0.1677 0.0378 0.2857 0.3226 0.0537 0.1428 0.1200	Draws  181  590  10,732  187  148  5,475	Distinct Ballots  181  590  10,710  187	Draws 94 295	Distinct Ballots 94 295	Draws 0.5193	R BRAVO size Distinct Ballots 0.5193
Alaska Arizona Arkansas California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.1677 0.0378 0.2857 0.3226 0.0537 0.1428	181 590 10,732 187 148	181 590 10,710 187	94 295	94	0.5193	
Alaska Arizona Arkansas California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.1677 0.0378 0.2857 0.3226 0.0537 0.1428	590 10,732 187 148	590 10,710 187	295			0.5193
Arizona Arkansas California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.0378 0.2857 0.3226 0.0537 0.1428	10,732 187 148	10,710 187		205		
Arkansas California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.2857 0.3226 0.0537 0.1428	187 148	187		493	0.5000	0.5000
California Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.3226 0.0537 0.1428	148		5,204	5,199	0.4849	0.4854
Colorado Connecticut Delaware DistrictOfColumbia Florida Georgia	0.0537 0.1428			96	96	0.5134	0.5134
Connecticut Delaware DistrictOfColumbia Florida Georgia	0.1428	5,475	148	79	79	0.5338	0.5338
Delaware DistrictOfColumbia Florida Georgia			5,470	2,676	2,675	0.4888	0.4890
DistrictOfColumbia Florida Georgia	0.1200	748	748	374	374	0.5000	0.5000
Florida Georgia	0.1200	1,057	1,056	523	523	0.4948	0.4953
Florida Georgia	0.9139	15	15	8	8	0.5333	0.5333
Georgia	0.0124	96,608	96,115	46,563	46,449	0.4820	0.4833
	0.0532	5,266	5,263	2,567	2,567	0.4875	0.4877
11411411	0.3488	128	128	68	68	0.5312	0.5312
Idaho	0.3662	120	120	64	64	0.5333	0.5333
Illinois	0.1804	474	474	242	242	0.5105	0.5105
Indiana	0.2023	374	374	187	187	0.5000	0.5000
Iowa	0.1013	1,520	1,520	753	753	0.4954	0.4954
Kansas	0.1013	318	318	162	162	0.5094	0.4934
	0.2222			79	79	0.5094	0.5094
Kentucky		155	155				
Louisiana	0.2034	365	365	182	182	0.4986	0.4986
Maine	0.0319	15,202	15,049	7,358	7,322	0.4840	0.4865
Maryland	0.2803	197	197	98	98	0.4975	0.4975
Massachusetts	0.2930	180	180	93	93	0.5167	0.5167
Michigan	0.0024	2,618,926	2,018,381	1,259,688	1,107,933	0.4810	0.5489
Minnesota	0.0166	56,680	56,139	27,421	27,294	0.4838	0.4862
Mississippi	0.1818	453	453	224	224	0.4945	0.4945
Missouri	0.1964	401	401	201	201	0.5012	0.5012
Montana	0.2222	320	320	164	164	0.5125	0.5125
Nebraska	0.2710	213	213	110	110	0.5164	0.5164
Nevada	0.0259	22,943	22,711	11,110	11,056	0.4842	0.4868
NewHampshire	0.0039	1,007,590	552,067	475,357	351,311	0.4718	0.6364
NewJersey	0.1457	703	703	350	350	0.4979	0.4979
NewMexico	0.0930	1,888	1,886	934	934	0.4947	0.4952
NewYork	0.2354	272	272	140	140	0.5147	0.5147
NorthCarolina	0.0381	10,330	10,319	5,000	4,998	0.4840	0.4843
NorthDakota	0.3962	98	98	55	55	0.5612	0.5612
Ohio	0.0854	2,077	2,077	1,018	1,018	0.4901	0.4901
Oklahoma	0.3861	101	101	55	55	0.5446	0.5446
Oregon	0.1231	1,068	1,068	535	535	0.5009	0.5009
Pennsylvania	0.0075	265,245	259,621	127,792	126,477	0.4818	0.4872
RhodeIsland	0.1662	562	562	280	280	0.4982	0.4982
SouthCarolina	0.1492	683	683	344	344	0.5037	0.5037
SouthDakota	0.3194	154	154	79	79	0.5130	0.5130
Tennessee	0.2725	206	206	106	106	0.5146	0.5146
Texas	0.0943	1,706	1,706	833	833	0.4883	0.4883
Utah	0.2477	329	329	165	165	0.5015	0.5015
Vermont	0.3037	180	180	91	91	0.5056	0.5056
Virginia	0.0565	4,790	4,788	2,329	2,329	0.4862	0.4864
Washington	0.1757	525	525	265	265	0.5048	0.5048
WestVirginia	0.4432	76	76	41	41	0.5395	0.5395
Wisconsin	0.0082	229,503	220,878	110,622	108,592	0.4820	0.3393
Wyoming	0.0082	59	220,878	110,622	108,392	0.4820	0.4916

Table 2: Comparison of *end-of-round* (EoR) Bravo and Athena First-Round Sizes for Statewide 2016 US Presidential Contests, for  $\delta=1.0$  and a stopping probability of 0.9, contd.

State	Margin	EoR Bravo		А	ATHENA	ATHENA size as a fraction	
					ı	l	BRAVO size
		Draws	Distinct Ballots	Draws	Distinct Ballots	Draws	Distinct Ballots
Alabama	0.2875	122	122	94	94	0.7705	0.7705
Alaska	0.1677	396	396	295	295	0.7449	0.7449
Arizona	0.0378	7,227	7,217	5,204	5,199	0.7201	0.7204
Arkansas	0.2857	128	128	96	96	0.7500	0.7500
California	0.3226	99	99	79	79	0.7980	0.7980
Colorado	0.0537	3,687	3,685	2,676	2,675	0.7258	0.7259
Connecticut	0.1428	502	502	374	374	0.7450	0.7450
Delaware	0.1200	716	716	523	523	0.7304	0.7304
DistrictOfColumbia	0.9139	10	10	8	8	0.8000	0.8000
Florida	0.0124	65,051	64,827	46,563	46,449	0.7158	0.7165
Georgia	0.0532	3,555	3,554	2,567	2,567	0.7221	0.7223
Hawaii	0.3488	86	86	68	68	0.7907	0.7907
Idaho	0.3662	83	83	64	64	0.7711	0.7711
Illinois	0.1804	318	318	242	242	0.7610	0.7610
Indiana	0.2023	254	254	187	187	0.7362	0.7362
Iowa	0.1013	1,024	1,024	753	753	0.7354	0.7354
Kansas	0.2222	215	215	162	162	0.7535	0.7535
Kentucky	0.3134	104	104	79	79	0.7596	0.7596
Louisiana	0.2034	247	247	182	182	0.7368	0.7368
Maine	0.0319	10,238	10,169	7,358	7,322	0.7187	0.7200
Maryland	0.2803	132	132	98	98	0.7424	0.7424
Massachusetts	0.2930	122	122	93	93	0.7623	0.7623
Michigan	0.2930	122	122	1,259,688	1,107,933	0.7023	0.7023
Minnesota	0.0024	38,185	37,939		27,294	0.7181	0.7194
		302	37,939	27,421	27,294		0.7194
Mississippi	0.1818					0.7417	
Missouri	0.1964	267	267	201	201	0.7528	0.7528
Montana	0.2222	217	217	164	164	0.7558	0.7558
Nebraska	0.2710	144	144	110	110	0.7639	0.7639
Nevada	0.0259	15,462	15,357	11,110	11,056	0.7185	0.7199
NewHampshire	0.0039	-	-	475,357	351,311	-	-
NewJersey	0.1457	478	478	350	350	0.7322	0.7322
NewMexico	0.0930	1,276	1,275	934	934	0.7320	0.7325
NewYork	0.2354	186	186	140	140	0.7527	0.7527
NorthCarolina	0.0381	6,961	6,956	5,000	4,998	0.7183	0.7185
NorthDakota	0.3962	70	70	55	55	0.7857	0.7857
Ohio	0.0854	1,403	1,403	1,018	1,018	0.7256	0.7256
Oklahoma	0.3861	69	69	55	55	0.7971	0.7971
Oregon	0.1231	724	724	535	535	0.7390	0.7390
Pennsylvania	0.0075	-	-	127,792	126,477	-	-
RhodeIsland	0.1662	382	382	280	280	0.7330	0.7330
SouthCarolina	0.1492	460	460	344	344	0.7478	0.7478
SouthDakota	0.3194	102	102	79	79	0.7745	0.7745
Tennessee	0.2725	138	138	106	106	0.7681	0.7681
Texas	0.0943	1,150	1,150	833	833	0.7243	0.7243
Utah	0.2477	220	220	165	165	0.7500	0.7500
Vermont	0.3037	122	122	91	91	0.7459	0.7459
Virginia	0.0565	3,229	3,228	2,329	2,329	0.7213	0.7215
Washington	0.1757	355	355	265	265	0.7465	0.7465
WestVirginia	0.4432	51	51	41	41	0.8039	0.8039
Wisconsin	0.0082	-	-	110,622	108,592	-	-
Wyoming	0.5141	40	40	29	29	0.7250	0.7250

Table 3: Comparison of selection-ordered (SB) Bravo and Athena First-Round Sizes for Statewide 2016 US Presidential Contests, for  $\delta=1.0$  and a stopping probability of 0.9, contd.

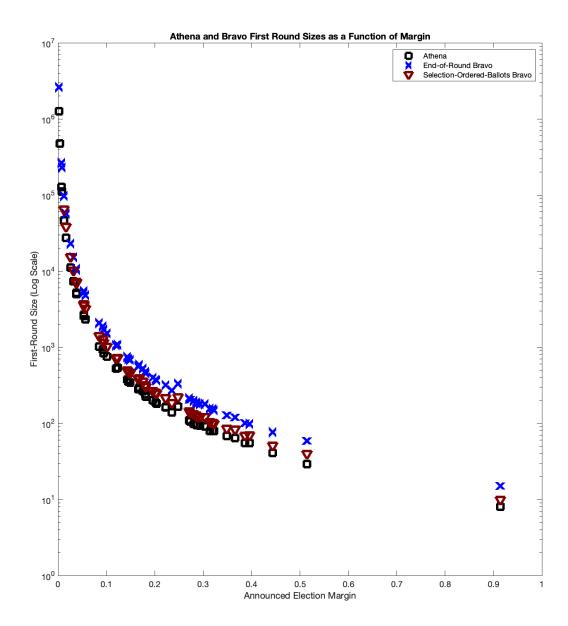


Figure 8: First-Round Sizes for 90% stopping probability: *End-of-Round BRAVO*, *Selection-Ordered-Ballots BRAVO* and ATHENA as a function of statewide margins of the 2016 US Presidential contest.

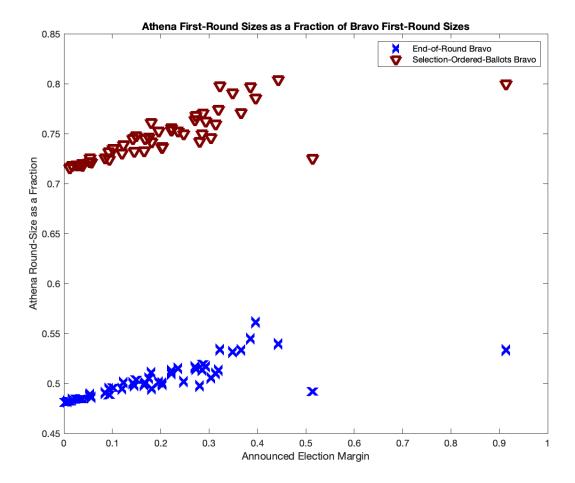


Figure 9: First-Round Sizes: ATHENA first-round sizes for 90% stopping probability as a fraction of those of *End-of-Round* BRAVO and *Selection-Ordered-Ballots*, for the statewide margins of the 2016 US Presidential contest.

#### **8.3** First-round Simulations

We observed that the stopping conditions ( $k_{min}$  values) for both ATHENA ( $\delta=1$ ) and MINERVA are identical for the ATHENA first round sizes presented in Table 2. This is because, for round sizes with large MINERVA stopping probabilities, the value of k is a very good representative of the underlying distribution, and the likelihood ratio for  $k \geq k_{min}$  is larger than 1.

We performed 100,000 simulations of MINERVA for each of the round sizes and corresponding margins (except for a couple of the low margin states, Michigan and New Hampshire); the results are presented in Table 4. The simulations used the declared tallies, and hence included ballots that were not votes for the two main candidates.

We observed that the empirical stopping probabilities for each state were slightly larger than 90%. Additionally, we observed that the risk of the first round for each state was smaller than 9%, and hence that the stopping probability to risk ratio was larger than  $\alpha^{-1}=10$ , which is as required by the stopping condition for MINERVA. Future work will include larger-scale and more complete simulations, as the risk-limited nature of the audit would need to be verified over multiple audit rounds.

State	Margin	Round Risk	Round Stopping Probability
Alabama	0.2875	0.0776	0.9152
Alaska	0.1677	0.0817	0.9067
Arizona	0.0378	0.0888	0.9019
Arkansas	0.2857	0.0790	0.9107
California	0.3226	0.0730	0.9167
Colorado	0.0537	0.0876	0.9009
Connecticut	0.1428	0.0821	0.9049
Delaware	0.1200	0.0844	0.9036
District Of Columbia	0.9139	0.0416	0.9478
Florida	0.0124	0.0886	0.9007
Georgia	0.0532	0.0870	0.9018
Hawaii	0.3488	0.0728	0.9156
Idaho	0.3662	0.0761	0.9144
Illinois	0.1804	0.0791	0.9097
Indiana	0.2023	0.0811	0.9083
Iowa	0.1013	0.0823	0.9051
Kansas	0.2222	0.0777	0.9074
Kentucky	0.3134	0.0748	0.9085
Louisiana	0.2034	0.0818	0.9074
Maine	0.0319	0.0896	0.9010
Maryland	0.2803	0.0800	0.9082
Massachusetts	0.2930	0.0736	0.9041
Michigan	0.0024	-	-
Minnesota	0.0166	0.0894	0.9008
Mississippi	0.1818	0.0836	0.9078
Missouri	0.1964	0.0793	0.9067
Montana	0.2222	0.0769	0.9080
Nebraska	0.2710	0.0739	0.9088
Nevada	0.0259	0.0881	0.9006
New Hampshire	0.0039	-	-
New Jersey	0.1457	0.0842	0.9034
New Mexico	0.0930	0.0852	0.9039
New York	0.2354	0.0771	0.9075
North Carolina	0.0381	0.0874	0.9018
North Dakota	0.3962	0.0705	0.9192
Ohio	0.0854	0.0858	0.9043
Oklahoma	0.3861	0.0732	0.9210
Oregon	0.1231	0.0830	0.9050
Pennsylvania	0.0075	0.0896	0.9006
Rhode Island	0.1662	0.0811	0.9050
South Carolina	0.1492	0.0813	0.9065
South Dakota	0.3194	0.0725	0.9097
Tennessee	0.2725	0.0752	0.9091
Texas	0.0943	0.0861	0.9021
Utah	0.2477	0.0774	0.9084
Vermont	0.3037	0.0788	0.9076
Virginia	0.0565	0.0879	0.901
Washington	0.1757	0.0812	0.9079
West Virginia	0.4432	0.0645	0.9126
Wisconsin	0.0082	0.089	0.9006
Wyoming	0.5141	0.0712	0.9039

Table 4: MINERVA Simulation Results for First-Round Sizes for Statewide 2016 US Presidential Contests; stopping probability of 0.9 and  $\alpha=0.1$ .

Figures 10, 11 and 12 present the stopping probability, the risk and the ratio of stopping probability to risk as a function of margin for all states except DC, which has a very large margin. Larger margins have very small round sizes, and the difference of a few ballots makes greater impact. This explains the points at margins of 0.5141 (Wyoming) and 0.4432 (West Virginia).

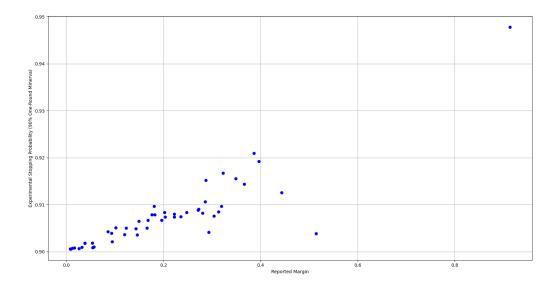


Figure 10: Simulation Results: Stopping Probability for Predicted ATHENA First Rounds for the 2016 US Presidential contest, Table 2.

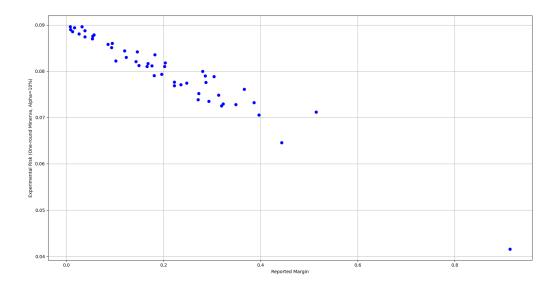


Figure 11: Simulation Results: Risk for Predicted ATHENA First Rounds for the 2016 US Presidential contest, Table 2.

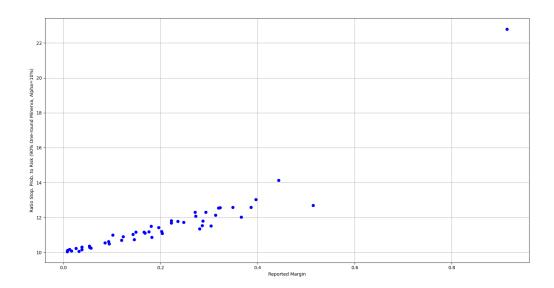


Figure 12: Simulation Results: Ratio of Stopping Probability to Risk for Predicted ATHENA First Rounds for the 2016 US Presidential contest, Table 2.

# 9 Conclusion

We describe inefficiencies with the use of audits developed for ballot-by-ballot decisions in round-to-round procedures, such as are in use in real audits today. We propose new audits, MINERVA and ATHENA, which we prove are risk-limiting and at least as efficient as audits that apply the ballot-by-ballot decision rules at the end of the round.

We describe an approach to computing stopping probabilities and risks of audits with stopping conditions that are monotone increasing with the number of ballots for the winner in the sample. We demonstrate its accuracy in reproducing the empirically-obtained percentile values from [5, Table 1], and find that the average fractional discrepancy is 0.13%.

We predict first round sizes (for 90% stopping probability) for all states in the US Presidential election of 2016 for end-of-round Bravo and Athena ( $\delta=1$ ). We find that our proposed audits require half the ballots across all margins. We similarly compare first round sizes to ordered-ballot-draw Bravo as well, finding 15-29% improvements, with the larger improvements corresponding to smaller margins. We thus see that the additional effort of retaining information on ballot order, required by selection-ordered-ballots Bravo, is not beneficial as the Athena class of audits does not require it.

We hope to present a third audit of the ATHENA class, METIS, which is more efficient for multiple-round audits, in a future draft of this manuscript.

Large first-round sizes for polling audits of low margin contests should not deter election officials from performing audits. Other options exist besides those reported in Section 8, which presents results for ballot polling audits only. Ballot comparison audits are far more efficient in terms of number of ballots needed for the audit; if cast vote records (CVRs) which can be efficiently matched with the corresponding paper ballot are easily available, or their creation requires less effort than the random sampling of a large number of ballots, they should be considered, especially for

low margin contests. It might also be possible to perform a combination of ballot polling audits and ballot comparison audits—such as described by Ottoboni *et al.* in the paper on *SUITE* [8]—to reduce effort.

We provide open-source software for computing probability distributions and for the MINERVA and ATHENA audits, hoping it can help developers of election auditing software. We also hope our work can help election officials planning audits.

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# A Proofs

#### A.1 Preliminaries

Before we prove the Theorems, we need the following general results from basic algebra.

**Lemma 2.** Given a monotone increasing sequence:  $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$ , for  $a_i, b_i > 0$ , the sequence:

$$z_i = \frac{\sum_{j=i}^n a_j}{\sum_{j=i}^n b_j}$$

is also monotone increasing.

*Proof.* Note that  $z_i$  is a weighted average of the values of  $\frac{a_j}{b_i}$  for  $j \geq i$ :

$$z_i = \sum_{j=i}^n y_j \frac{a_j}{b_j}$$

$$y_j = \frac{b_j}{\sum_{j=i}^n b_j} > 0$$

$$\sum_{j=i}^{n} y_j = 1 \Rightarrow y_j \le 1$$

$$y_i = 1 \Leftrightarrow i = j = n$$

Observe that, because  $\frac{a_i}{b_i}$  is monotone increasing,

$$z_i \ge \frac{a_i}{b_i}$$

with equality if and only if i = n. Suppose i < n. Then:

$$z_{i+1} \ge \frac{a_{i+1}}{b_{i+1}} > \frac{a_i}{b_i}$$

$$z_i = y_i \frac{a_i}{b_i} + (1 - y_i)z_{i+1} < z_{i+1}$$

And  $z_i$  is also monotone increasing.

**Lemma 3.** Given a strictly monotone increasing sequence:  $x_1, x_2, \dots x_n$  and some constant A,

$$\exists i_{min} \ such \ that \ x_i \geq A \Leftrightarrow i \geq i_{min}$$

*Proof.* Let  $i_{min}$  be the first index for which the sequence exceeds or equals A. That is, let  $i_{min}$  be such that

$$x_{i_{min}} \ge A \ x_i < A \ 1 \le j < i_{min}$$

Because the sequence is monotone increasing,

$$x_i > x_{i_{min}} \ge A \ \forall i > i_{min}$$

If no elements in the sequence exceed or equal A, let  $i_{min} = n + 1$ .

**Lemma 4.** Given p, n, with  $p > \frac{1}{2}$ ,  $\sigma(k, p, n)$  is strictly monotone increasing as a function of k.

Proof.

$$p > \frac{1}{2} \Rightarrow p > 1 - p \Rightarrow \frac{1 - p}{p} < 1$$

Let  $0 \le k < n$ . Then:

$$\sigma(k,p,n) = \frac{p^k(1-p)^{n-k}}{(\frac{1}{2})^n} = \frac{1-p}{p} \cdot \frac{p^{k+1}(1-p)^{n-(k+1)}}{(\frac{1}{2})^n} = \frac{1-p}{p}\sigma(k+1,p,n)$$

Hence

$$\sigma(k, p, n) < \sigma(k + 1, p, n) \ \forall \ k \ such \ that \ 0 \le k < n$$

#### A.2 Proofs of properties of the complementary cdf ratios

**Theorem 1.** For the  $(\alpha, p, (n_1, n_2, ..., n_j, ...))$ -MINERVA test, if the round schedule is pre-determined (before the audit begins), the following are true for j = 1, 2, 3, ...

1.

$$\frac{s_j(k_j)}{r_j(k_j)} = \sigma(k_j, p, n_j)$$

when  $r_i(k_i)$  and  $s_i(k_i)$  are defined and non-zero.

- 2.  $\tau_j(k_j, p, (n_1, n_2, \dots, n_j), \alpha)$  is strictly monotone increasing as a function of  $k_j$ .
- 3.  $\exists k_{min,j}(MINERVA, (n_1, n_2, \dots, n_j), p, \alpha)$  such that

$$\mathcal{A}(X_j) = Correct \Leftrightarrow K_j \geq k_{min,j}(\text{Minerva}, (n_1, n_2, \dots, n_j), p, \alpha)$$

*Proof.* We show this by induction.

Consider j = 1.

1.

$$\frac{s_1(k_1)}{r_1(k_1)} = \frac{Pr[K_1 = k_1 \mid H_a, n_1]}{Pr[K_1 = k_1 \mid tie, n_1]} = \sigma(k_1, p, n_1)$$

2.

$$\tau_1(k_1, p, n_1) = \frac{Pr[K_1 \ge k_1 \mid H_a, n_1]}{Pr[K_1 \ge k_1 \mid tie, n_1]} = \frac{S_1(k_1)}{R_1(k_1)} = \frac{\sum_{k=k_1}^{k_{max,1}} s_1(k)}{\sum_{k=1}^{n} r_1(k)}$$

where  $k_{max,j}$  is the largest possible value for  $k_j$ . Note that  $k_{max,1} = n_1$ .

is a weighted average of  $\sigma(k, p, n_1)$ , and, by Lemmas 2 and 4, is strictly monotone increasing as a function of  $k_1$ .

3. From Lemma 3,  $\exists k_{min,1}(MINERVA, (n_1), p, \alpha)$  such that

$$au_1(k_1,p,n_1) \geq rac{1}{lpha} \Leftrightarrow k_1 \geq k_{min,1}(\text{Minerva},(n_1),p,lpha)$$

which is the Minerva stopping condition.

Thus the theorem is true for j = 1.

Suppose the theorem is true for j = m. We will show it is true for j = m + 1.

From property (3) of this theorem for j=m, we observe that, after the stopping decision is made and before the next round is drawn, the number of winner ballots in the sample is strictly smaller than  $k_{min,m}(\text{MINERVA}, (n_1, n_2, \dots, n_m), p, \alpha)$ . The distribution on the winner votes may be modeled as  $s_m^*(k_m)$  and  $r_m^*(k_m)$  where:

$$s_m^*(k_m) = \begin{cases} s_m(k_m) & k < k_{min,m}(\text{MINERVA}, (n_1, n_2, \dots, n_m), p, \alpha) \\ 0 & else \end{cases}$$

and

$$r_m^*(k_m) = \begin{cases} r_m(k_m) & k < k_{min,m}(\text{MINERVA}, (n_1, n_2, \dots, n_m), p, \alpha) \\ 0 & else \end{cases}$$

When we draw the next round of ballots with replacement, the resulting distributions on the winner ballots are convolutions:

$$s_{m+1}(k_{m+1}) = s_m^*(k_m) \circledast binomial(k_{new,m+1}, p, n_{m+1} - n_m)$$

and

$$r_{m+1}(k_{m+1}) = r_m^*(k_m) \otimes binomial(k_{new,m+1}, 0.5, n_{m+1} - n_m)$$

where  $\circledast$  represents the convolution operator and binomial(k, p, n) the probability of drawing k ballots for the winner in a sample of size n from a distribution with fractional tally p for the winner. Using property (1) of this theorem for j=m, we see that

$$s_m^*(k_m) = A(k_m)p^{k_m}(1-p)^{n_m-k_m}$$

and

$$r_m^*(k_m) = A(k_m)(\frac{1}{2})^{n_m}$$

for some A, a function of k, current and previous round sizes, p and  $\alpha$ .

Some book keeping demonstrates that

$$s_{m+1}(k_{m+1}) = B(k_{m+1})p^{k_{m+1}}(1-p)^{n_{m+1}-k_{m+1}}$$

where

$$B(k_{m+1}) = A(k_m) \circledast \binom{n_{m+1} - n_m}{k_{new,m+1}}$$

And

$$r_{m+1}(k_{m+1}) = B(k_{m+1})(\frac{1}{2})^{n_{m+1}}$$

which proves property (1) for j = m + 1. Properties (2) and (3) follow for j = m + 1 by application of Lemmas 2-4.

Thus the theorem is true for all  $j \ge 1$ .

**Theorem 2.** For the  $(\alpha, \delta, p, (n_1, n_2, \dots, n_j, \dots))$ -ATHENA test, if the round schedule is pre-determined (before the audit begins), the following are true for  $j = 1, 2, 3, \dots$ :

1.

$$\frac{s_j(k_j)}{r_j(k_j)} = \sigma(k_j, p, n_j)$$

when  $r_i(k_i)$  and  $s_i(k_i)$  are defined and non-zero.

- 2.  $\omega_i(k_i, p, (n_1, n_2, \dots, n_i), \alpha, \delta)$  is strictly monotone increasing.
- 3.  $\exists k_{min,j}(ATHENA, (n_1, n_2, \dots, n_j), p, \alpha, \delta)$  such that

$$\mathcal{A}(X_j) = Correct \Leftrightarrow k_j \geq k_{min,j}(ATHENA, (n_1, n_2, \dots, n_j), p, \alpha)$$

*Proof.* The proof proceeds exactly as that for Theorem 1, except that there are two stopping conditions, which may be represented as:

$$\omega_j(k_j, p, (n_1, n_2, \dots, n_j)) \ge \frac{1}{\alpha} \Leftrightarrow k_j \ge k_{min,j}(\omega, (n_1, n_2, \dots, n_j), p, \alpha)$$

and

$$\sigma(k_j, p, n_j) \ge \frac{1}{\delta} \Leftrightarrow k_j \ge k_{min}(\mathsf{BRAVO}, n, p, \alpha)$$

Hence

$$k_{min,j}(\text{ATHENA}, (n_1, n_2, \dots, n_j), p, \alpha) = max(k_{min,j}(\omega, (n_1, n_2, \dots, n_j), p, \alpha), k_{min}(\text{Bravo}, n, p, \alpha))$$

A.3 Proof of risk-limiting property of MINERVA

**Theorem 3.**  $(\alpha, H_a, (n_1, n_2, \dots, n_j, n_{j+1}, n_{j+2}, \dots))$ -MINERVA is an  $\alpha$ -RLA if the round schedule is pre-determined (before the audit begins).

Proof. From Definition 6 and Theorem 1, we have

$$R_j = Pr[K_j \ge k_{min,j}(\text{Minerva}, (n_1, n_2, \dots, n_j), p, \alpha) \mid H_0, n_j]$$

$$\le \alpha \cdot Pr[K_j \ge k_{min,j}(\text{Minerva}, (n_1, n_2, \dots, n_j), p, \alpha) \mid H_a, n_j]$$

$$= \alpha \cdot S_i$$

because  $k_{min,j}(\text{MINERVA},(n_1,n_2,\ldots,n_j),p,\alpha)$  satisfies the MINERVA stopping condition.

Define the total stopping probability of the audit as follows:

$$S = Pr[(\mathcal{A}(X) = Correct) \mid H_a]$$

Then,

$$S = \sum_{j} S_{j} \le 1 \tag{11}$$

The risk of the audit is defined as:

$$R = Pr[(\mathcal{A}(X) = Correct) \mid H_0] = \sum_{j} R_j \le \alpha \cdot \sum_{j} S_j = \alpha \cdot S \le \alpha$$

from Equation (11).

## A.4 Properties of B2 versions of MINERVA and ATHENA

**Theorem 5.** The B2  $(\alpha, p)$ -Bravo audit stops for a sample of size  $n_j$  with  $k_j$  ballots for the winner, if and only if the  $(\alpha, H_a, (1, 2, 3, \ldots, j, \ldots))$ -MINERVA audit stops.

*Proof.* Consider the  $j^{th}$  round of the MINERVA audit: the  $j^{th}$  ballot draw. Suppose that, before the  $j^{th}$  round is drawn, and after the stopping condition is tested for the  $(j-1)^{th}$  round and the audit stopped if it is satisfied, k is the largest value of winner ballots possible. It is strictly smaller than the corresponding  $k_{min,j-1}$ , because the audit has stopped for all other values. Further, because at most one winner ballot will be drawn in the  $j^{th}$  round, the largest possible number of winner ballots in the  $j^{th}$  round is k+1.

More formally, let the largest value of  $k_{j-1}$  for which  $s_{j-1}^*(k_{j-1}) \neq 0$  be k, where  $s_j^*$  is as defined in the proof of Theorem 1. Then

$$k < k_{min,j-1}(MINERVA, (n_1, n_2, \dots, n_{j-1}), p, \alpha)$$

by the definition of  $k_{min,j-1}$ , Theorem 1. Further, the largest value of  $k_j$  for which  $s_j(k_j) \neq 0$  is k+1.

We now show that if the  $j^{th}$  round stops at all, it will be for  $k_j = k + 1$  and no other values of  $k_j$ .

We observe that the only way to obtain k+1 ballots in the  $j^{th}$  round is if the existing number of winner ballots is k and the new ballot drawn is for the winner. The probability is:

$$s_j(k+1) = ps_{j-1}(k)$$

On the other hand, k ballots arise in the  $j^{th}$  round if the existing number is k-1 and a winner ballot is drawn, or the existing number is k and the ballot drawn is not for the winner.

$$s_{i}(k) = (1-p)s_{i-1}(k) + ps_{i-1}(k-1)$$

Similarly:

$$r_j(k+1) = \frac{1}{2}r_{j-1}(k)$$

and

$$r_j(k) = \frac{1}{2}r_{j-1}(k) + \frac{1}{2}r_{j-1}(k-1)$$

If the condition is satisfied by values other than k+1, because  $\tau$  is monotone increasing, it is satisfied by k:

$$\tau_j(k) = \frac{s_j(k+1) + s_j(k)}{r_j(k+1) + r_j(k)} = \frac{s_{j-1}(k) + ps_{j-1}(k-1)}{r_{j-1}(k) + \frac{1}{2}r_{j-1}(k-1)} \ge \frac{1}{\alpha}$$

Thus  $au_j(k)$  is a weighted average of  $\sigma(k,p,j-1)$  and  $rac{p}{\frac{1}{2}}\sigma(k-1,p,j-1)$  and:

$$\frac{p}{\frac{1}{2}}\sigma(k-1,p,j-1) = \frac{(1-p)}{\frac{1}{2}}\sigma(k,p,j-1) < \sigma(k,p,j-1) < \tau(k,p,j-1) < \frac{1}{\alpha}$$

as  $k < k_{min,j-1}(\text{MINERVA}, (1,2,\ldots,j-1), p, \alpha)$ . And hence,  $\tau_j(k)$  does not pass the stopping condition.

Thus, if  $A_M$  and  $A_B$  denote the B2 MINERVA and B2 BRAVO audits respectively,

$$\mathcal{A}_M(X_j) = Correct \Leftrightarrow \tau_j(k, p, j) \geq \frac{1}{\alpha} \Leftrightarrow \sigma(k, p, j) \geq \frac{1}{\alpha} \Leftrightarrow \mathcal{A}_B(X_j) = Correct$$

Samples that do satisfy the stopping condition have the same MINERVA and BRAVO p-values, which are otherwise not the same.

**Corollary 1.** Given  $\alpha, p, \delta$  such that  $\delta \geq \alpha$ , the B2  $(\alpha, p)$ -Bravo audit stops for a sample of size  $n_j$  with  $k_j$  ballots for the winner, if and only if the  $(\alpha, \delta, H_a, (1, 2, 3, \ldots, j, \ldots))$ -Athena audit stops.

*Proof.* As in Theorem 5, the  $j^{th}$  audit round stops only for the largest possible number of winner votes if it does at all. Thus, it stops if and only if it satisfies the  $(\alpha, p)$ -BRAVO stopping condition. Additionally, the second stopping condition for ATHENA is also a BRAVO condition, and is always satisfied when the first one is satisfied because  $\delta \geq \alpha$ .

## A.5 Proof of efficiency property of MINERVA and ATHENA

**Theorem 6.** Given sample X of size  $n_j$  with  $k_j$  samples for the winner,

$$A_B(X) = Correct \Rightarrow A_A(X) = Correct$$

where  $A_B$  denotes the  $(\alpha, p)$ -Bravo test and  $A_A$  the  $(\alpha, p, (n_1, n_2, \ldots, n_j, n_{j+1}, n_{j+2}, \ldots))$ -Minerva test or the  $(\alpha, \delta, p, (n_1, n_2, \ldots, n_j, n_{j+1}, n_{j+2}, \ldots))$ -Athena test for  $\delta \geq \alpha$  if the round schedule is pre-determined (before the audit begins).

*Proof.* Note that for a fixed election and fixed round sizes, each of  $\tau$  and  $\omega$ , the complementary cdf stopping conditions for MINERVA and ATHENA respectively, is a weighted sum of  $\sigma$ , the monotone increasing BRAVO stopping condition. Further, if k is the number of winner ballots, the elements in the weighted sum are at least as large as  $\sigma$ . In fact, equality for  $\tau$  occurs only when k is the largest possible number of winner ballots in the round. Thus

$$\sigma(k_j, p, n_j) \ge \frac{1}{\alpha} \Rightarrow \tau_j(k_j, p, (n_1, n_2, \dots, n_j)), \omega_j(k_j, p, (n_1, n_2, \dots, n_j, \dots)) \ge \frac{1}{\alpha}$$

Note that ATHENA has a second condition, which is also satisfied

$$\sigma(k_j, p, n_j) \ge \frac{1}{\alpha} \Rightarrow \sigma(k_j, p, n_j) \ge \frac{1}{\delta}$$